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> restart; with(LinearAlgebra) :
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Generating the example

Example for the paper

```
> S6 := Matrix(6, 6, [x·z, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, x^2, 0, 0, 0, 0, 0, 0, x·z, 0, 0, 0, 0, 0, 0,
    1, 0, 0, 0, 0, 0, 0, y]) : S5 := Matrix(6, 6, [1, 0, 0, 2·y, 0, 0, 0, 1, 0, 0, 1, y^(-2), 0, 0, 1, 0, 0,
    -1, 0, 3, 0, 1, 6, 5 + y^(-2), 0, 0, 0, 0, 1, 6 + y^(-2), 0, 0, 0, 0, 0, 1]) : S1 := Matrix(6, 6, [y, 0,
    0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, y^(-1)]) : S2
    := Matrix(6, 6, [1, y^(-3), 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1,
    0, 0, 0, 0, 0, 0, 1]) : S3 := Matrix(6, 6, [1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0,
    0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]) : S4 := Matrix(6, 6, [y^2, 0, 0, 0, 0, 0, 0, y^2, 0, 0, 0,
    0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, y, 0, 0, 0, 0, 0, 0, y]) : F
    := MatrixMatrixMultiply(S1, MatrixMatrixMultiply(S2, MatrixMatrixMultiply(S3, S4))) : F
    := MatrixMatrixMultiply(S5, F) : F := MatrixMatrixMultiply(S6, F) : A := map(simplify,
    MatrixMatrixMultiply(map(diff, F, x), MatrixInverse(F))) : B := map(simplify,
    MatrixMatrixMultiply(map(diff, F, y), MatrixInverse(F))) : C := map(simplify,
    MatrixMatrixMultiply(map(diff, F, z), MatrixInverse(F))) : T2 := Matrix(6, 6, [1, 0, 0, 0,
    z^2·y^2, x·z·y^2, 0, 1, z^2·y^4, 0, 0, 0, 0, 0, 1, x + y^3, z, 0, 0, 0, 0, 1, 3·x^3·y, 0, 0, 0, 0, 0, 1, y^2·x, 0,
    0, 0, 0, 0, 1]) : T1 := Matrix(6, 6, [y^3, 0, 0, 0, 0, 0, 0, y^3, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1,
    0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]) : T3 := Matrix(6, 6, [1, 0, 0, 0, 0, 0, x + y, 1, 0, 0, 0, 0,
    0, z^2·y, 1, 0, 0, 0, 0, x^4·z·3 + y, x^3·y + z, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 2, x^3, 1]) : T
    := MatrixMatrixMultiply(T3, MatrixMatrixMultiply(T1, T2)) : Tinv := MatrixInverse(T) :
    G := MatrixMatrixMultiply(T, F) : A1 := map(simplify, MatrixMatrixMultiply(map(diff,
    G, x), MatrixInverse(G))) : B1 := map(simplify, MatrixMatrixMultiply(map(diff, G, y),
    MatrixInverse(G))) : C1 := map(simplify, MatrixMatrixMultiply(map(diff, G, z),
    MatrixInverse(G))) :
```

Verification of the integrability conditions, the leading matrix coeff w.r.t. y, and its rank

```
> Equal(map(simplify, map(diff, A1, y) + MatrixMatrixMultiply(A1, B1)), map(simplify,
    map(diff, B1, x) + MatrixMatrixMultiply(B1, A1))); Equal(map(simplify, map(diff, B1, z)
    + MatrixMatrixMultiply(B1, C1)), map(simplify, map(diff, C1, y)
    + MatrixMatrixMultiply(C1, B1))); Equal(map(simplify, map(diff, A1, z)
    + MatrixMatrixMultiply(A1, C1)), map(simplify, map(diff, C1, x)
    + MatrixMatrixMultiply(C1, A1))); B10 := subs(y = 0, y^4·B1); Rank(%);
```

true

true

true

$B10 := [[0, 0, 0, 0, 0, 0],$

$[0, 0, 0, 0, 0, 0],$

$[48 x^7 z^2 + 18 x^5 z^2 - 6 x^3 z, -48 x^6 z^2 - 18 x^4 z^2 + 6 x^2 z, -16 x^2 z^2 - 8 x^2 z - 6 z^2$
$- 3 z, 16 x^2 z + 6 z, 8 x^5 z + 3 x^3 z, -8 x^2 z - 3 z],$

$[48 x^7 z^3 + 48 x^6 z^2 + 18 x^5 z^3 - 6 x^3 z^2 - 6 x^2 z, -48 x^6 z^3 - 48 x^5 z^2 - 18 x^4 z^3$
$+ 6 x^2 z^2 + 6 x z, -16 x^2 z^3 - 8 x^2 z^2 - 16 x z^2 - 6 z^3 - 8 x z - 3 z^2, 16 x^2 z^2 + 16 x z$

$$+ 6 z^2, 8 x^5 z^2 + 8 x^4 z + 3 x^3 z^2, -8 x^2 z^2 - 8 x z - 3 z^2],$$
$$[18 x^5 z, -18 x^4 z, -6 z - 3, 6, 3 x^3, -3],$$
$$[18 x^8 z - 48 x^7 z^2 + 96 x^6 z^2 - 18 x^5 z^2 + 6 x^3 z - 12 x^2 z, -18 x^7 z + 48 x^6 z^2$$
$$- 96 x^5 z^2 + 18 x^4 z^2 - 6 x^2 z + 12 x z, -6 x^3 z + 16 x^2 z^2 - 3 x^3 + 8 x^2 z - 32 x z^2$$
$$- 16 x z + 6 z^2 + 3 z, 6 x^3 - 16 x^2 z + 32 x z - 6 z, 3 x^6 - 8 x^5 z + 16 x^4 z - 3 x^3 z,$$
$$-3 x^3 + 8 x^2 z - 16 x z + 3 z]]$$

$$2 \qquad \textbf{(1)}$$

T0*T1 is a unimodular column reduction transformation (with a permutation) and T2 is a shearing in y

```
> T0 := Matrix(6, 6, [1, 0, 0, 0, 0, 0, x, 1, 0, 0, 0 , 0, 0, 0, 1, 0, 0, 0, 0, 0, z
  + 1/2, 1, -1/2*x^3, 1/2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]) : T1 := Matrix(6, 6, [0, 0, 0, 1, 0,
    0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]) : T2
    := Matrix(6, 6, [y, 0, 0, 0, 0, 0, 0, y, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0,
    0, 0, 0, 0, 0, 1]) : T3 := MatrixMatrixMultiply(T0, MatrixMatrixMultiply(T1, T2));
    Determinant(T0); A2 := map(simplify, MatrixMatrixMultiply(MatrixInverse(T3),
    MatrixMatrixMultiply(A1, T3) - map(diff, T3, x))) : C2 := map(simplify,
    MatrixMatrixMultiply(MatrixInverse(T3), MatrixMatrixMultiply(C1, T3) - map(diff, T3,
    z))) : B2 := map(simplify, MatrixMatrixMultiply(MatrixInverse(T3),
    MatrixMatrixMultiply(B1, T3) - map(diff, T3, y))) :
```

$$T3 := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & y & 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ y & 0 & z + \frac{1}{2} & 0 & -\frac{1}{2} x^3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$1 \qquad \textbf{(2)}$$

Poincare rank dropped by one. And now B21 is the leading matrix of the new leading coefficient and which is of rank 2

```
> map(ldegree, B2, y); B20 := map(coeff, B2, y, -3); Rank(B21);
```

$$\begin{bmatrix} -3 & -3 & -3 & -3 & -1 & 2 \\ -1 & -2 & -1 & -2 & 0 & 3 \\ -3 & -3 & -3 & -3 & -1 & 2 \\ 0 & -2 & 0 & -2 & 0 & 3 \\ -3 & -3 & -3 & -3 & -1 & \infty \\ -3 & -3 & -3 & -3 & -1 & 2 \end{bmatrix}$$

$$B20 := [[16 x^5 z + 6 x^3 z, -48 x^9 z^2 - 18 x^7 z^2 + 6 x^5 z, -16 x^8 z - 6 x^6 z, 48 x^9 z^2 + 18 x^7 z^2$$
$$- 6 x^5 z, 0, 0],$$

$[0, 0, 0, 0, 0, 0],$

$[16 x^2 z + 6 z, -48 x^6 z^2 - 18 x^4 z^2 + 6 x^2 z, -16 x^5 z - 6 x^3 z, 48 x^6 z^2 + 18 x^4 z^2 - 6 x^2 z, 0, 0],$

$[0, 0, 0, 0, 0, 0],$

$[6, -18 x^4 z, -6 x^3, 18 x^4 z, 0, 0],$

$[6 x^3 - 16 x^2 z + 32 x z - 6 z, -18 x^7 z + 48 x^6 z^2 - 96 x^5 z^2 + 18 x^4 z^2 - 6 x^2 z + 12 x z, -6 x^6 + 16 x^5 z - 32 x^4 z + 6 x^3 z, 18 x^7 z - 48 x^6 z^2 + 96 x^5 z^2 - 18 x^4 z^2 + 6 x^2 z - 12 x z, 0, 0]]$

$$2 \qquad \textbf{(3)}$$

Verification of the normal crossings

> $map(ldegree, C2, y); map(ldegree, A2, y);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 7 & 7 & 6 & 0 & 5 & 6 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 8 & 8 & 8 & 0 & 5 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 3 \\ 7 & 7 & 6 & 0 & 5 & 6 \\ 3 & 2 & 0 & 2 & 0 & 3 \\ 8 & 8 & 8 & 0 & 5 & \infty \\ 3 & 3 & 3 & 3 & \infty & \infty \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \textbf{(4)}$$

The product of the transformations to be performed to arrive finally at a system of first kind (we know from the construction that the ssytem is regular)

> $T7 := Matrix(6, 6, [x^3, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]) : T8 := Matrix(6, 6, [0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]) : T2 := Matrix(6, 6, [y^2, 0, 0, 0, 0, 0, 0, y^2, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]) : T9 := MatrixMatrixMultiply(T7, MatrixMatrixMultiply(T8, T2)); A3 := map(simplify, MatrixMatrixMultiply(MatrixInverse(T9), MatrixMatrixMultiply(A2, T9) - map(diff, T9, x))) : C3 := map(simplify, MatrixMatrixMultiply(MatrixInverse(T9), MatrixMatrixMultiply(C2, T9) - map(diff, T9, z))) : B3 := map(simplify, MatrixMatrixMultiply(MatrixInverse(T9), MatrixMatrixMultiply(B2, T9) - map(diff, T9, y))) :$

$$T9 := \begin{bmatrix} y^2 & 0 & x^3 & 0 & 0 & 0 \\ 0 & y^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Verification of the normal crossings

> $map(ldegree, C3, y); map(ldegree, A3, y); map(ldegree, B3, y);$

$$\begin{bmatrix} 7 & 0 & 4 & \infty & 4 & 4 \\ 7 & 7 & 4 & \infty & 4 & 4 \\ 3 & 3 & 0 & \infty & 0 & 0 \\ 10 & 10 & \infty & 0 & 5 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty \\ 3 & 3 & 0 & \infty & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 4 & \infty & 4 & 4 \\ 7 & 7 & 4 & \infty & 4 & 4 \\ 5 & 4 & 0 & \infty & 0 & 3 \\ 10 & 10 & \infty & 0 & 5 & \infty \\ 5 & 5 & \infty & \infty & \infty & \infty \\ 3 & 3 & 0 & \infty & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 3 & \infty & -1 & 3 \\ -1 & -1 & 3 & \infty & -1 & 3 \\ -1 & -1 & 2 & \infty & -1 & 2 \\ 2 & 0 & 3 & -1 & 0 & 3 \\ -1 & -1 & \infty & \infty & -1 & \infty \\ -1 & -1 & 2 & \infty & -1 & 2 \end{bmatrix} \tag{6}$$

The Poincare rank in y dropped again but by two this time. And now the Poincare rank is (zero, zero, zero).

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