Formal Solutions of Completely Integrable Pfaffian Systems With Normal Crossings

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Abstract

In this paper, we present an algorithm for computing a fundamental matrix of formal solutions of completely integrable Pfaffian systems with normal crossings in several variables. This algorithm is a generalization of a method developed for the bivariate case based on a combination of several reduction techniques and is partially [1] implemented in the computer algebra system MAPLE.

Key words: Linear systems of partial differential equations, Pfaffian systems, Formal solutions, Rank reduction, Hukuhara-Turrittin's normal form, Normal crossings.

1. Introduction

Pfaffian systems arise in many applications [20], including the studies of aerospace, celestial mechanics [16], and statistics [24]. So far, the most important systems for applications are those with so-called normal crossings [31].

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[1] Our Maple package PfaffInt can be downloaded at: `http://www.mjaroschek.com/pfaffian/` It contains functionalities illustrated by examples for the splitting, column reduction, rank reduction, and the computation of exponential parts of multivariate completely integrable systems with normal crossings

A univariate completely integrable Pfaffian system with normal crossings reduces to a singular linear system of ordinary differential equations (ODS, in short), which have been studied extensively (see [5, 34] and references therein). Moreover, unlike the general case of several variables considered herein, algorithms to related problems leading to the computation of formal solutions have been developed by various authors (see [6, 11, 12, 13] and references therein). The MAPLE package ISOLDE [14] and MATHEMAGIX package LINDALG [28] are dedicated to the symbolic resolution of such systems.

More recently, bivariate systems were treated by the first and third author of this paper in [1]. This paper refines the results of the bivariate case and generalizes them to treat the more general multivariate case.

To get an intuition of the kind of systems we consider, we informally study the following simple bivariate completely integrable Pfaffian system with normal crossings. A formal definition of these systems will be given in Section 2.

Example 1. [1, Example 2] Given the following bivariate system over the ring of formal power series in (x_1, x_2) with complex coefficients:

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} F = A_1 F = \begin{pmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{pmatrix} F, \\ x_2^3 \frac{\partial}{\partial x_2} F = A_2 F = \begin{pmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{pmatrix} F, \end{cases}$$

we are interested in constructing the formal objects F that satisfy the system. The existence of a fundamental matrix of solutions and its general form follows from well known theoretical results (see Corollary 6). The proof, however, is not constructive. For simplicity, we assume we already know that a fundamental matrix of formal solutions in our particular case is of the form

$$\Phi(x_1, x_2) x_1^{C_1} x_2^{C_2} e^{q_1(x_1^{-1/s_1})} e^{q_2(x_2^{-1/s_2})}, \tag{1}$$

where $\Phi(x_1, x_2)$ is a matrix with formal power series entries, C_1 and C_2 are matrices with entries in $\mathbb{C}$, and q_1, q_2 are polynomials in $\mathbb{C}[z_1], \mathbb{C}[z_2]$ respectively. We now want to determine $\Phi, C_1, C_2, q_1, q_2, s_1, s_2$. For this purpose, we use the algorithm presented in [1]. The main idea is to compute one part of the solution by considering an associated ODS in only one variable and then use this information to compute the other parts of the solution by transforming and decoupling the system into smaller and simpler systems:

• First, we construct two associated systems whose equations are derived by setting either $x_1 = 0$ or $x_2 = 0$:

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} F = A_1(x_1, 0) \, F = \begin{pmatrix} x_1^3 + x_1^2 & 0 \\ -1 & x_1^3 + x_1^2 \end{pmatrix} F, \\ x_2^3 \frac{\partial}{\partial x_2} F = A_2(0, x_2) \, F = \begin{pmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{pmatrix} F. \end{cases}$$

We show in Section 4 that the formal invariants q_1, q_2, s_1 and s_2 can be computed from these associated systems. Via ISOLDE or LINDALG we compute $s_1 = s_2 = 1$ and $q_1(1/x_1) = \frac{-1}{x_1}$, $q_2(1/x_2) = (\frac{3}{x_2^2} + \frac{2}{x_2})$, and (1) becomes

$$\Phi(x_1, x_2) x_1^{C_1} x_2^{C_2} e^{\frac{-1}{x_1}} e^{\frac{3}{x_2^2} + \frac{2}{x_2}}.$$

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- Next, we apply the so-called eigenvalue shifting $F = e^{\frac{-1}{x_1}} e^{\frac{3}{x_2^2} + \frac{2}{x_2}} G$ (for a new unknown vector G), to facilitate the next step. The shifting yields:

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} G = \begin{pmatrix} x_1^3 + x_2 & x_2^2 \\ -1 & x_1^3 - x_2 \end{pmatrix} G, \\ x_2^2 \frac{\partial}{\partial x_2} G = \begin{pmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{pmatrix} G. \end{cases}$$

- After the eigenvalue-shifting we apply another transformation that reduces the orders of the singularities in x_1 and x_2 to their minimal integer values. By setting $G = T_1 H$ where

$$T_1 = \begin{pmatrix} x_2 x_1^3 & -x_2 \\ 0 & 1 \end{pmatrix},$$

we get:

$$\begin{cases} x_1 \frac{\partial}{\partial x_1} H = \begin{pmatrix} -2 & 0 \\ -x_2 & 1 \end{pmatrix} H, \\ x_2 \frac{\partial}{\partial x_2} H = \begin{pmatrix} -2 & 0 \\ -2x_1^3 & -1 \end{pmatrix} H. \end{cases}$$

- Finally, via some linear algebra (see [25, Chapter 3] for general cases) we compute the transformation

$$T_2 = \begin{pmatrix} 1 & 0 \\ \frac{x_2}{3} + 2x_1^3 & -1 \end{pmatrix},$$

and setting $H = T_2 U$ results in the system

$$\begin{cases} x_1 \frac{\partial}{\partial x_1} U = C_1 U = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} U, \\ x_2 \frac{\partial}{\partial x_2} U = C_2 U = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} U. \end{cases}$$

We can now read off C_1 and C_2. We collect the applied transformations and get a fundamental matrix of solutions:

$$\underbrace{T_1 T_2}_{=: \Phi} x_1^{C_1} x_2^{C_2} e^{\frac{-1}{x_1}} e^{\frac{3}{x_2^2} + \frac{2}{x_2}},$$

where $C_1 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ and $C_2 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$.

Unlike this simple example, the steps of computation can be far more involved and demand multiple levels of recursion. In order to generalize this algorithm to more than two variables, the following nontrivial questions have to be addressed:

- Can the information on the formal invariants still be obtained from the associated ODS systems?

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- Can a rank reduction algorithm be developed without relying on properties of principal ideal domains as in the univariate and bivariate case?

The results of [1] which have an immediate generalization to the multivariate setting are refined herein and supported by fully transparent proofs and illustrating figures (Theorem 13 which answers positively and unconditionally the first question, the structure of the main algorithm described in Section 3, and Theorem 18). However, the answer to the second question is more elaborate (see Section 5) and requires the discussion of two problems which are not discussed in [1]:

- The major obstacle to a generalization of the results on the bivariate case lies in the process of finding integral relations among generators of certain modules over the ring of multivariate power series. In Section 5.4, we propose a solution that relaxes the condition of working over a principal ideal domain to working over local rings and show how to utilize Nakayama's Lemma in the formal reduction process if the modules under investigation are free.
- We discuss an algorithmic difficulty which arises as not all formal power series under manipulation admit a finite representation, even if the input Pfaffian system is given in a finite form (see Example 23). Although this problem arises in the bivariate case as well, it has not been addressed before (neither in [1] nor in [15, 25]). We provide a reasoning to check the correctness of our algorithm.

We thus present the first comprehensive description of the state of the art algorithmic approach for solving completely integrable Pfaffian systems with normal crossings in the multivariate setting. Our investigation also involves the multivariate versions of the transformations used classically in the well-studied univariate case (e.g. shearing transformations in Section 5.3, column reduction transformations in Section 5.4, and properties of transformations in Proposition 9). Not only does this discussion serve the manipulation of such transformations within our proposed formal reduction, but it also plays a role in future generalizations of many other algorithms available for univariate systems (e.g. the alternative rank reduction algorithm of Section 6.2 and the notion of *simple systems* as suggested in the conclusion).

   This paper is divided as follows: In Section 2, we recall the basic definitions and the necessary theory for our algorithm. This includes the general form of the solutions, the notion of equivalence between systems, the classification of singularities, and a description of the necessary transformations whose generalization to the multivariate case is straightforward. In Section 3, we give the general structure of our proposed algorithm which relies on two major components: The first is associating to our system a set of ODS's from which its formal invariants can be efficiently derived. This is detailed in Section 4. The second component is the rank reduction which we give in Section 5. The main algorithm is then given in Section 6 before concluding in Section 7.

2. Preliminaries

2.1. *Completely Integrable Pfaffian Systems with Normal Crossings*

   The systems considered in this paper are those whose associated differential form is a 1-form. More explicitly, let $R := \mathbb{C}[[x_1, \ldots, x_n]]$ be the ring of formal power series

in $x_1, x_2, \ldots, x_n$ over the field of complex numbers $\mathbb{C}$. A Pfaffian system with normal crossings is a system of linear partial differential equations of the form

$$x_i^{p_i+1} \frac{\partial}{\partial x_i} F = A_i F, \quad 1 \leq i \leq n, \tag{2}$$

where the A_i's are $d \times d$ matrices with entries in R. The system (2) is completely determined by the A_i's and p_i's and we conveniently denote it by $[A]$. Each of the p_i's is an integer and the number $p_i' := \max(0, p_i)$ is called the Poincaré rank of the ith component A_i. The n-tuple

$$p := (p_1', \ldots, p_n')$$

is called the Poincaré rank of the system $[A]$. If $p_i \leq -1$ for every $i \in \{1, \ldots, n\}$ then the origin is an ordinary (non-singular) point of the system and the system is said to be regular. In this paper, we tackle the rather more interesting singular systems. The singular locus of a system with normal crossings is a union of hyperplanes of coordinates $x_1 x_2 \ldots x_n = 0$. A Pfaffian system is called completely integrable, if the following commutation rule holds for all $i, j \in \{1, \ldots, n\}$:

$$A_i A_j - A_j A_i = x_i^{p_i+1} \frac{\partial}{\partial x_i} A_j - x_j^{p_j+1} \frac{\partial}{\partial x_j} A_i. \tag{3}$$

Subsequently, whenever we refer to a Pfaffian system, we assume it is a completely integrable system with normal crossings. For the remainder of this paper we once and for all fix a Pfaffian system $[A]$ for which all the A_i are non-zero and there is at least one strictly positive p_i. All subsequent definitions and theorems are stated in this setting, disregarding systems for which the origin is an ordinary point.

2.2. Notations and Algebraic Structures

Our notations follow a set of guidelines in order to help the reader remember the multitude of different objects involved in our work. Single letter identifiers are usually chosen to be the initial letter of the mathematical term attached to the referenced object, like d for dimension and R for a ring. For a vector v, its ith component is given by v_i and for a univariate power series s the ith coefficient is denoted by s_i. We do not distinguish between row and column vectors. Upper case letters are used for algebraic structures, matrices and the unknown in a Pfaffian system. A family of matrices is given with lower indices, e.g. $(M_{i,j})_{i,j \geq 0}$, and for a matrix $M_{i,j}$, blocks are given with upper indices, e.g.

$$M_{i,j} = \begin{pmatrix} M_{i,j}^{11} & M_{i,j}^{12} \\ M_{i,j}^{21} & M_{i,j}^{22} \end{pmatrix},$$

where the size of the different blocks are clear from the context. By x we denote the collection of variables $x_1, \ldots, x_n$ and we use $\bar{x}_i$ to refer to the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$.

One can expand the A_i in system $[A]$ as a formal power series with respect to x_i:

$$x_i^{p_i+1} \frac{\partial}{\partial x_i} F = (A_{i,0} + A_{i,1} x_i + A_{i,2} x_i^2 + \ldots) F,$$

where the $A_{i,j}$ are elements of $\mathbb{C}[[\bar{x}_i]]$. We denote this ring by $R_{\bar{x}_i}$. The first coefficient $A_{i,0} = A(x_i = 0)$ in such an expansion can be regarded as non-zero without any loss of

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generality, otherwise p_i can be readjusted. We call $A_{i,0}$ the leading matrix coefficient of the ith component.

Aside from the rings R and $R_{\bar{x}_i}$, we will frequently have to work in other algebraic structures. We denote by $K := \text{Frac}(R)$ (respectively $K_{\bar{x}_i}$) the fraction field of R (respectively $R_{\bar{x}_i}$). Let L be the set of monomials given by,

$$L = \{x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n} \text{ for } \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n\}.$$

Clearly, L is closed under multiplication and contains the unit element. Then, one can define $R_L := L^{-1}R$, the localization of R at L, i.e. the ring of series with only finitely many terms having monomials of strictly negative exponents. Unlike in the univariate case, K and R_L do not refer to the same algebraic structure, e.g. $(x_1 + x_2)^{-1}$ is an element of K which is not an element of R_L. In fact, there exists no $\beta \in \mathbb{N}^2$ such that $x^\beta(x_1+x_2)^{-1} \in R$: $(x_1+x_2)^{-1} = \frac{1/x_1}{1-(-x_2/x_1)}$ whose formal expansion with respect to x_2 is $\sum_{i\geq 0}(-1)^i x_1^{-i-1} x_2^i$, which has infinitely many poles in x_1. For a further characterization of K, one may refer to [3].

Finally, in the sequel it will be necessary to introduce ramifications of the form $x_i = t_i^{\alpha_i}$ for new variables t_i and positive integers α_i. We will therefore write R_t for $\mathbb{C}[[t_1, \ldots, t_n]]$ and allow analogous notations for all structures introduced so far. The identity and zero matrices of dimension d are denoted by I_d and O_d, and we set: $\text{GL}_d(R) = \{M \in \text{Mat}_{d\times d}(R) \mid \det(M) \text{ is invertible in R}\}$.

2.3. Equivalent Systems

As we have seen in the introductory example, we will make use of transformations to bring a system into particular forms. Such a transformation acts on a Pfaffian system as follows: A linear transformation (also called gauge transformation) $F = TG$, where $T \in GL_d(K)$, applied to (2) results in the system

$$x_i^{\tilde{p}_i+1} \frac{\partial}{\partial x_i} G = \tilde{A}_i G, \quad 1 \leq i \leq n, \tag{4}$$

where

$$\frac{\tilde{A}_i}{x_i^{\tilde{p}_i+1}} = T^{-1}(\frac{A_i}{x_i^{p_i+1}} T - \frac{\partial}{\partial x_i} T), \quad 1 \leq i \leq n. \tag{5}$$

We say that system (4) is *equivalent* to system (2) and we write $T[A] := [\tilde{A}]$. It can be easily verified that complete integrability is inherited by an equivalent system. Subsequently, to stay in the same class of systems under study, special care will be taken so that the transformations used in our considerations do not alter the normal crossings. In fact, a major difficulty within the symbolic manipulation of system (2) arises from (5). It is evident that any transformation alters all the components simultaneously. In particular, the equivalent system does not necessarily inherit the normal crossings even for very simple examples.

Example 2. [15, Section 4] Consider the following completely integrable Pfaffian system with normal crossings of Poincaré rank $(3, 1)$:

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} F = A_1(x_1, x_2)\, F = \begin{pmatrix} x_1^3 + x_2 & x_2^2 \\ -1 & -x_2 + x_1^3 \end{pmatrix} F, \\[2mm] x_2^2 \frac{\partial}{\partial x_2} F = A_2(x_1, x_2)\, F = \begin{pmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{pmatrix} F. \end{cases}$$

This system appears within the reduction of the system of Example 1 in the introduction. As we have seen, there exists a transformation which drops p_1 to zero. This can also be attained by the transformation

$$F = \begin{pmatrix} x_1^3 & -x_2^2 \\ 0 & x_2 \end{pmatrix} G,$$

which is computed by the univariate-case Moser-based rank reduction algorithm, upon regarding the first component as an ODS in x_1 and x_2 as a transcendental constant. This results in the equivalent system

$$\begin{cases} x_1 x_2 \frac{\partial}{\partial x_1} G = \tilde{A}_1(x_1, x_2)\, G = \begin{pmatrix} -2x_2 & 0 \\ -1 & x_2 \end{pmatrix} G, \\[2mm] x_2^3 \frac{\partial}{\partial x_2} G = \tilde{A}_2(x_1, x_2)\, G = \begin{pmatrix} -x_2^2 & 0 \\ -2x_1^3 & -2x_2^2 \end{pmatrix} G. \end{cases}$$

We can see that such a transformation achieves the goal of reducing the Poincaré rank of the first component. However, it alters the normal crossings as it introduces the factor x_2 on the left hand side of the first component. Moreover, it elevates the Poincaré rank of the second component.

In order to preserve the normal crossings, we restrict the class of transformations that we use in our algorithm:

Definition 3. Let $T \in GL_d(\mathrm{K})$. We say that the transformation $F = TG$ (respectively T) is *weakly compatible* with system $[A]$ if $T[A] := \tilde{A}$ is again a completely integrable Pfaffian system with normal crossings. In particular, $\tilde{A}_i \in \mathrm{R}^{d \times d}$ for every $i \in \{1, \ldots, n\}$.

Clearly, any constant or unimodular invertible matrix is an example of such transformations.

In the sequel, we will also need to resort to transformations with stronger properties:

Definition 4. Let $T \in GL_d(\mathrm{K})$. We say that the transformation $F = TG$ (respectively T) is *compatible* with system $[A]$ if it is weakly compatible with $[A]$ and the Poincaré rank of each individual component of $T[A]$ does not exceed that of the respective component of $[A]$.

2.4. *Fundamental Matrix of Formal Solutions*

Before studying how to construct formal solutions to a given system, the question arises if and how many solutions exist. The language of stable modules over the ring

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of power series is used in [18, Theorem 1] and [22, Main Theorem] independently to establish the following theorem which gives an answer to this question.

Theorem 5. There exist strictly positive integers α_i, $1 \le i \le n$, and an invertible matrix $T \in \mathrm{R}_t^{d \times d}$ such that, upon setting $x_i = t_i^{\alpha_i}$, the transformation $T(t)$ yields the following equivalent system:

$$t_i^{\alpha_i \tilde{p}_i + 1} \frac{\partial}{\partial t_i} G = \tilde{A}_i(t_i) G, \quad 1 \le i \le n,$$

where

$$\tilde{A}_i(t_i) = \mathrm{Diag}(\tilde{A}_i^{11}(t_i), \tilde{A}_i^{22}(t_i), \dots, \tilde{A}_i^{jj}(t_i)),$$

and for every $\ell \in \{1, \dots, j\}$ we have that $\tilde{A}_i^{\ell\ell}(t_i)$ is a square matrix of dimension d_ℓ of the form

$$\tilde{A}_i^{\ell\ell}(t_i) = w_i^{\ell\ell}(t_i) I_{d_\ell} + t_i^{\alpha_i \tilde{p}_i}(c_i^{\ell\ell} I_{d_\ell} + N_i^{\ell\ell}),$$

where
- $d_1 + d_2 + \cdots + d_j = d$;
- $w_i^{\ell\ell}(t_i) = \sum_{m=0}^{\alpha_i \tilde{p}_i - 1} \lambda_{im\ell} t_i^m$ is a polynomial in t_i, with coefficients in $\mathbb{C}$;
- $c_i^{\ell\ell} \in \mathbb{C}$ and $N_i^{\ell\ell}$ is a constant (with respect to all derivations $\partial/\partial t_i$) d_ℓ-square matrix having nilpotent upper triangular form;
- for any fixed $\ell \in \{1, \dots, j\}$, the matrices $\{N_i^{\ell\ell}\}_{i=1,\dots,n}$ are permutables;
- for all $\ell \in \{1, \dots, j-1\}$, there exists $i \in \{1, \dots, n\}$ such that

$$w_i^{\ell\ell}(t_i) \ne w_i^{(\ell+1)(\ell+1)}(t_i) \qquad \text{or} \qquad c_i^{\ell\ell} - c_i^{(\ell+1)(\ell+1)} \notin \mathbb{Z}.$$

This theorem guarantees the existence of a transformation which takes system (2) to the so-called Hukuhara-Turrittin's normal form from which the construction of a fundamental matrix of formal solutions (6) is straightforward. In fact, we have:

Corollary 6. Given system (2), a fundamental matrix of formal solutions exists and is of the form

$$\Phi(x_1^{1/s_1}, \dots, x_n^{1/s_n}) \prod_{i=1}^n x_i^{C_i} \prod_{i=1}^n \exp(Q_i(x_i^{-1/s_i})), \tag{6}$$

where Φ is an invertible matrix with entries in R_t and for each $i \in \{1, \dots, n\}$ we have:
- s_i is a positive integer;
- the diagonal matrix

$$Q_i(x_i^{-1/s_i}) = \mathrm{Diag}\left(q_{i,1}(x_i^{-1/s_i}), q_{i,2}(x_i^{-1/s_i}), \dots, q_{i,d}(x_i^{-1/s_i})\right)$$

contains polynomials in x_i^{-1/s_i} over $\mathbb{C}$ without constant terms. We refer to $Q_i(x_i^{-1/s_i})$ as the x_i-exponential part. Under the notations of Theorem 5, it is obtained by formally integrating $\frac{w_i^{\ell\ell}}{t_i^{\alpha_i \tilde{p}_i + 1}}$;
- C_i is a constant matrix which commutes with $Q_i(x_i^{-1/s_i})$.

A singular system $[A]$ is said to be *regular singular* whenever, for every $i \in \{1, \dots, d\}$, $Q_i(x_i^{-1/s_i})$ is a zero matrix. Otherwise, system (2) is said to be *irregular singular* and the entries of $Q_i(x_i^{-1/s_i})$, $1 \le i \le n$, determine the main asymptotic behavior of the actual solutions as $x_i \to 0$ in appropriately small sectorial regions [23, Proposition 5.2, pp 232, and Section 4].

Definition 7. Let $i \in \{1, \ldots, n\}$. If $Q_i(x_i^{-1/s_i})$ is a nonzero matrix then we set $m_{i,j}$ to be the minimum order in x_i within the terms of $q_{i,j}(x_i^{-1/s_i})$ for $1 \leq j \leq d$. The x_i-*formal exponential growth order* (x_i-exponential order, in short) of A_i is the rational number

$$\omega(A_i) = -\min_{1 \leq j \leq d} m_{i,j}.$$

The n-tuple of rational numbers $\omega(A) = (\omega(A_1), \ldots, \omega(A_n))$ then defines the exponential order of system $[A]$. Otherwise, we set $\omega(A_i) = 0$.

If two systems are equivalent then they have the same x_i-exponential parts, and consequently the same x_i exponential orders, for all $1 \leq i \leq n$, under any transformation $T \in GL_d(\mathrm{K})$.

Example 8 (Example 1 cont.)**.** From our investigations in the example of Section 1, we see that for the given fundamental system of formal solutions, we have non-zero exponential parts with $\omega(A_1) = 1$ and $\omega(A_2) = 2$ and so the system is irregular singular (although $s_1 = s_2 = 1$).

The above theoretical results on existence do not establish the formal reduction itself, that is the algorithmic procedure which computes explicitly the α_i's and a transformation which takes the system to a normal form that allows the construction of such solutions. This will be our interest in the following sections.

The computation of the formal invariants is a difficult task in the univariate case [6, 9]. However, we will prove in Section 4 that in the multivariate case, these invariants can be computed from associated univariate systems. Unlike the univariate case, the main difficulties of the algorithm lie in rank reduction. Before proceeding to describe the algorithms we propose, we give a property of the transformations which can be deployed:

Proposition 9. Consider a completely integrable Pfaffian system $[A]$ with normal crossings. Let $T \in GL_d(\mathrm{K})$ and set $T[A] = [\tilde{A}]$. If T is a transformation which is weakly compatible with system $[A]$ then $T \in GL_d(R_L)$.

Proof. It follows from (5) that

$$\frac{\partial}{\partial x_i} T = \frac{A_i}{x_i^{p_i+1}} T - T \frac{\tilde{A}_i}{x_i^{\tilde{p}_i+1}}, \quad 1 \leq i \leq n.$$

Thus, we have (see, e.g. [5, Proposition 1, proof, pp 6]):

$$\frac{\partial}{\partial x_i} \det(T) = \left(\frac{\mathrm{tr}(A_i)}{x^{p_i+1}} - \frac{\mathrm{tr}(\tilde{A}_i)}{x^{\tilde{p}_i+1}} \right) \det(T), \quad 1 \leq i \leq n. \tag{7}$$

Therefore, $\det(T)$ itself is a solution of a completely integrable Pfaffian system with normal crossings. By Corollary 6, $\det(T)$ has the form (6). Since $T \in \mathrm{K}^{d \times d}$ then $\det(T)$ is free of logarithmic and exponential terms. Hence, $\det(T)$ corresponds to a log-free regular solution of (7). Thus, $\det(T) \in \mathrm{R}_L$. The same argument serves to prove that $\det(T^{-1})$ is an element of R_L as well, upon remarking that $T^{-1}[\tilde{A}] = [A]$. Hence, $\det(T)^{-1} \in \mathrm{R}_L$ and consequently $T \in R_L^{d \times d}$. $\square$

However, the converse of Proposition (9) is not true, which complicates the task of constructing adequate transformations in the reduction process (see, e.g., Example 2 or the shearing transformations of Section 5.3).
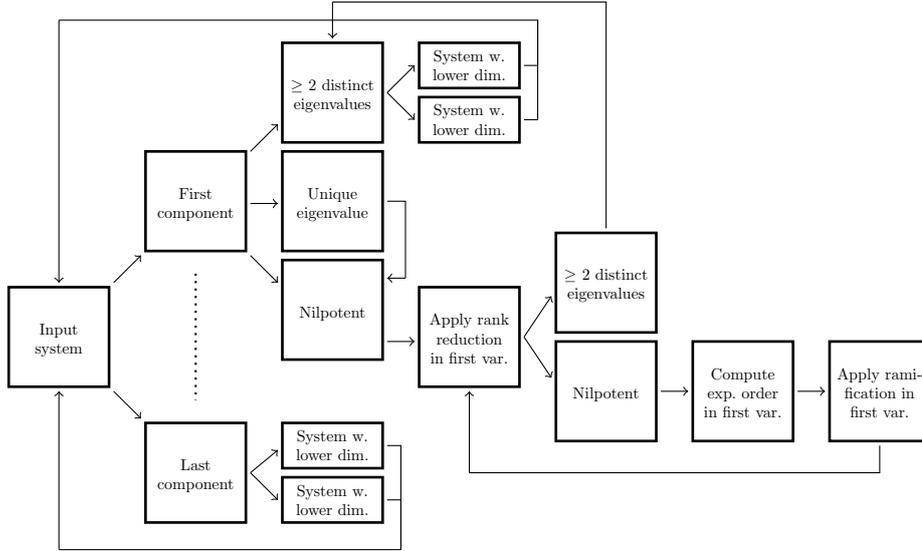
Fig. 1. Computing a fundamental matrix of formal solutions by working with one of the components, e.g. the first component. The other components would follow the chosen component in the uncoupling.

3. Structure of the Main Algorithm

If one is only interested in the asymptotic behavior of the solutions of system $[A]$, then one can compute the formal invariants from associated univariate systems as we prove in Section 4. If the singularity is regular or one is interested in computing a full fundamental matrix of formal solutions, as given by (6), then, besides computing these invariants, further involved steps are required, as illustrated in Example 1. The recursive algorithm we propose generalizes that of the univariate case given by the first author in [6]. At each level of recursion with input $[A]$, we consider the leading matrix coefficients $A_{i,0} = A_i(x_i = 0)$ (we use both notation interchangeably) and distinguish between three main cases:

(1) There exists at least one index $i \in \{1, \ldots, n\}$ such that $A_{i,0}$ has at least two distinct eigenvalues.
(2) All of the leading matrix coefficients have exactly one eigenvalue and there exists at least one index $i \in \{1, \ldots, n\}$ such that $A_{i,0}$ has a nonzero eigenvalue.
(3) For all $i \in \{1, \ldots, n\}$, $A_{i,0}$ is nilpotent.

In order to identify the properties of the eigenvalues of $A_i(x_i = 0)$, it suffices to consider the constant matrix $A_i(x = 0)$ due to the following well-known proposition (see, e.g., [22, Proposition 1, pp 8] or [8, Proposition 2.2] for a proof within the context of eigenrings):

Proposition 10. The eigenvalues of $A_{i,0}$, $1 \leq i \leq n$, belong to $\mathbb{C}$.

Then, based on the above classification, a linear or an exponential transformation will be computed as described in the following subsections.

3.1. Distinct Eigenvalues: Uncoupling the System Into Systems of Lower Dimensions

Whenever there exists an index $i \in \{1, \ldots, n\}$ such that $A_{i,0}$ has at least two distinct eigenvalues, the system can be uncoupled into subsystems of lower dimensions as shown in Theorem 11. For a constructive proof, one may refer to [23, Section 5.2, pp 233].

Theorem 11. Suppose that for some $i \in \{1, \ldots, n\}$, the leading matrix coefficient $A_{i,0}$ has at least two distinct eigenvalues. Then there exists a unique transformation $T \in GL_d(\mathrm{R})$ of the form

$$T(x) = \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix} = \begin{pmatrix} I_{d'} & T^{12}(x) \\ T^{21}(x) & I_{d-d'} \end{pmatrix},$$

where $0 < d' < d$, such that the transformation $F = TG$ yields the equivalent system

$$x_i^{p_i+1} \frac{\partial}{\partial x_i} G = \begin{pmatrix} \tilde{A}_i^{11}(x) & O \\ O & \tilde{A}_i^{22}(x) \end{pmatrix} G, \quad 1 \leq i \leq n.$$

and $\tilde{A}_i^{11}(x), \tilde{A}_i^{22}(x)$, $i \in \{1, \ldots, n\}$ are of dimensions d' and $d - d'$ respectively.

The theorem can be restated by saying that if one of the components of the system has a leading matrix coefficient with at least two distinct eigenvalues, then it can be uncoupled. All of the other components are uncoupled simultaneously. In the sequel, we aim to determine changes of the independent variables x_i (ramifications), and construct transformations, which will allow the reduction of any input system to a system for which the leading matrix coefficient of at least one of its components has at least two distinct eigenvalues. This allows us to either arrive at a system with lower Poincaré rank or uncouple it into several subsystems of lower dimensions. The recursion stops whenever we arrive at regular singular ($p = (0, \ldots, 0)$) or scalar ($d = 1$) subsystems. The former have been already investigated in [25, Chapter 3] and the resolution of the latter is straightforward.

We remark that, by Proposition 10, it suffices that there exists $i \in \{1, \ldots, n\}$ such that the constant matrix $A_i(x_1 = 0, \ldots, x_i = 0, \ldots, x_n = 0)$ has at least two distinct eigenvalues.

3.2. Unique Eigenvalue: Shifting

For any $i \in \{1, \ldots, n\}$ such that $A_{i,0}$ has a unique nonzero eigenvalue $\gamma_i \in \mathbb{C}$, applying the so-called eigenvalue shifting

$$F = \exp\left(\int^{x_i} \gamma_i z_i^{-p_i-1} dz_i\right) G,$$

yields a system $[\tilde{A}]$ whose i^{th} component has a nilpotent leading matrix coefficient:

$$x_i^{\tilde{p}_i+1} \frac{\partial}{\partial x_i} G = \tilde{A}_i(x) G, \quad \text{where} \quad \tilde{A}_i(x) = A_i(x) - \gamma_i I_d.$$

The other components of the system are not modified by this transformation which is clearly compatible with system $[A]$.

Hence, due to the uncoupling and shifting, we can assume without loss of generality that for all $i \in \{1, \ldots, n\}$, the leading matrix coefficients $A_{i,0}$ are nilpotent.

3.3. Nilpotency: Rank Reduction and Exponential Order

In the univariate case, $n = 1$, the nilpotency of $A_{1,0}$ suggests at least one of the following two steps, as proposed by the first author in [6]: Rank reduction and computation of the exponential order $\omega(A_1)$. The former reduces p_1 to its minimal integer value. It is possible that p_1 drops to zero, i.e. we arrive at a regular singular system, or that the leading matrix coefficient of the resulting system has at least two distinct eigenvalues, in which case we can again uncouple the system. Otherwise, $\omega(A_1) = \ell/m$ is to be computed, where ℓ and m are coprime. Then, by setting $x_1 = t_1^m$ and applying rank reduction again, it is proven that we arrive at a system whose leading matrix coefficient has two distinct eigenvalues. Therefore, the system can be uncoupled (see Figure 1).

The bivariate case, $n = 2$, is studied by the first and third authors of this paper in [1]. For rank reduction, the properties of principal ideal domains were used. To determine the formal exponential order $\omega(A)$, associated univariate systems were defined. In this paper, we show that on the one hand, this approach to determine the formal exponential order remains valid in the multivariate setting, as we will see in the next section. On the other hand, the generalization of the rank reduction algorithm to the multivariate case is nontrivial and is discussed in Section 5. The multivariate formal reduction algorithm is then summed up in Section 6.

4. Computing the Formal Invariants

In the univariate case, where the system is given by a single matrix A_1, $\omega(A_1)$ can be computed from the characteristic polynomial of A_1, i.e. $\det(\lambda I_d - A_1)$, based on the analysis of a Newton polygon associated with the system [6, Theorem 1]. In this section we show that one need not search for a generalization of this algorithm to the multivariate case as the formal invariants of $[A]$, i.e. the exponential parts and $\omega(A)$, can be obtained from an associated univariate system. We do not only give a method to retrieve these invariants but we also reduce computations to computations with univariate rather than multivariate formal series.

Definition 12. Given a Pfaffian system $[A]$, we call the following the *associated ODS* of $[A]$:
$$x_i^{p_i+1} \frac{d}{dx_i} \mathcal{F}_i = \mathcal{A}_i(x_i) \, \mathcal{F}_i, \quad 1 \le i \le n,$$
where $\quad \mathcal{A}_i(x_i) := A(x_1 = 0, \ldots, x_{i-1} = 0, x_i, x_{i+1} = 0, \ldots, x_n = 0).$

Theorem 13. For every $i \in \{1, \ldots, n\}$, the x_i-exponential part of a Pfaffian system is equal to the exponential part of the ith component of its associated ODS.

To establish this result, we rely on a triangular form weaker than the Hukuhara-Turrittin's normal form given in Theorem 5. This weaker form suffices to give insight into the computation of (6).

The following theorem is an reformulation of a theorem which was first given in [17, Proposition 3, pp 654] for the bivariate case, and then generalized in [18, Theorem 2.3] to the general multivariate case.
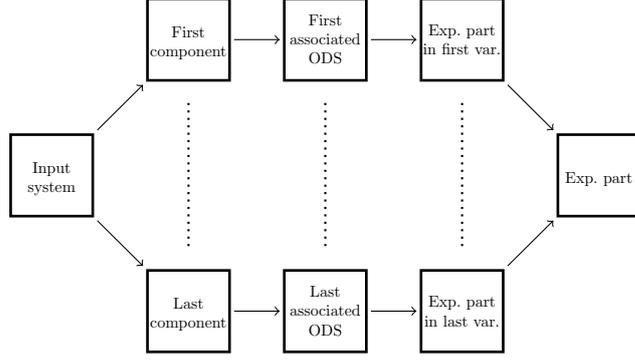
Fig. 2. Computing the exponential part from associated ODS's

Theorem 14. Consider the Pfaffian system $[A]$. There exists a positive integer α_1, and a transformation $T \in GL_d(\mathrm{K_t})$ (where $x_1 = t_1^{\alpha_1}$ and $x_i = t_i$, $2 \le i \le n$), such that the transformation $F = TG$ yields the equivalent system:

$$
\begin{cases}
t_1^{\alpha_1 \hat{p}_1 + 1} \frac{\partial}{\partial t_1} G = \hat{A}_1(t_1, x_2, \dots, x_n)\, G, \\
x_i^{\hat{p}_i + 1} \frac{\partial}{\partial x_i} G = \hat{A}_i(x_2, \dots, x_n)\, G, \quad 2 \le i \le n,
\end{cases}
\tag{8}
$$

where

$$
\begin{aligned}
\tilde{A}_1(t_1, x_2, \dots, x_n) &= \mathrm{Diag}(\hat{A}_1^{11}, \hat{A}_1^{22}, \dots, \hat{A}_1^{jj}), \\
\hat{A}_i(x_2, \dots, x_n) &= \mathrm{Diag}(\hat{A}_i^{11}, \hat{A}_i^{22}, \dots, \hat{A}_i^{jj}), \quad 2 \le i \le n,
\end{aligned}
$$

and for all $\ell \in \{1, \dots, j\}$ and $i \in \{2, \dots, n\}$ the entries of $\hat{A}_i^{\ell\ell}$ lie in $\mathrm{R}_{\bar{x}_1}$. The $\hat{A}_1^{\ell\ell}$'s are of the form

$$
\hat{A}_1^{\ell\ell} = w_1^{\ell\ell}(t_1) I_{d_\ell} + t_1^{\alpha_1 \hat{p}_1} (\hat{N}_1^{\ell\ell}(x_2, \dots, x_n) + c_1^{\ell\ell} I_{d_\ell}),
$$

where
- $d_1 + d_2 + \cdots + d_j = d$;
- $w_1^{\ell\ell}(t_1)$ and $c_1^{\ell\ell}$ are as in Theorem 5;
- If $\ell, \ell' \in \{1, \dots, j-1\}$ and $\ell \ne \ell'$, then $w_1^{\ell\ell}(t_1) \ne w_1^{\ell'\ell'}(t_1)$ or $c_1^{\ell\ell} - c_1^{\ell'\ell'} \notin \mathbb{Z}$;
- $\hat{N}_1^{\ell\ell}(x_2, \dots, x_n)$ is a nilpotent d_ℓ-square matrix whose entries lie in $\mathrm{R}_{\bar{x}_1}$.

Moreover, T can be chosen as a product of transformations in $GL_d(\mathrm{R_t})$ and transformations of the form $\mathrm{Diag}(t_1^{\beta_1}, \dots, t_1^{\beta_d})$, where $\beta_1, \dots, \beta_d$ are non-negative integers. $\qquad \square$

Proof of Theorem 13. Upon the change of independent variable $x_1 = t_1^{\alpha_1}$, the transformation $F = TG$ yields system (8) for which the first component is given by

$$
t_1^{\alpha_1 \hat{p}_1 + 1} \frac{\partial}{\partial t_1} G = \tilde{A}_1(t_1, x_2, \dots, x_n)\, G.
$$

with the notations and properties as in Theorem 14. It then follows from (5) that

$$
t_1^{\alpha_1 \hat{p}_1 + 1} \frac{\partial}{\partial t_1} T = \alpha_1 A_1(x_1 = t_1^{\alpha_1})\, T - T \hat{A}_1.
\tag{9}
$$

Due to the particular choice of T in Theorem 14, we can set $x_i = 0$, $2 \le i \le n$ in (9). In

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particular, the relation between the leading terms

$$
\begin{aligned}
\mathcal{A}_1(x_1 = t_1^{\alpha_1}) &:= A_1(x_1 = t_1^{\alpha_1}, x_2 = 0, \ldots, x_n = 0), \\
\hat{\mathcal{A}}_1(t_1) &:= \hat{A}_1(t_1, x_2 = 0, \ldots, x_n = 0), \\
\mathcal{T}(t_1) &:= T(t_1, x_2 = 0, \ldots, x_n = 0),
\end{aligned}
$$

is given by

$$
t_1^{\alpha_1 \hat{p}_1 + 1} \frac{\partial}{\partial t_1} \mathcal{T} = \alpha_1 \mathcal{A}_1(x_1 = t_1^{\alpha_1}) \, \mathcal{T} - \mathcal{T} \hat{\mathcal{A}}_1.
$$

Hence, the systems given by $\alpha_1 \mathcal{A}_1(x_1 = t_1^{\alpha_1})$ (respectively $\mathcal{A}_1$) and $\hat{\mathcal{A}}_1$ are equivalent. It follows that they have the same formal invariants. Clearly, the same result can be obtained for any of the other components via permutation with the first component. Noting that the x_i-exponential part is independent of $\bar{x}_i$ completes the proof. □

For univariate systems, the *true Poincaré rank* $p_{true}(A_1)$ is defined as the smallest integer greater or equal than the exponential order $\omega(A_1)$ of the system $[A_1]$. It is known that this integer coincides with the minimal value for p_1 which can be obtained upon applying any non-ramified linear transformation to A_1. With the help of Theorem 13 we can establish the analogous result for multivariate systems. We first give the following definition:

Definition 15. Let $[A]$ be a Pfaffian system and for any $i \in \{1, \ldots, n\}$, let $p_{true}(A_i)$ be the minimal integer value which bounds the exponential order in x_i, i.e.

$$
p_{true}(A_i) - 1 < \omega(A_i) \leq p_{true}(A_i).
$$

Then $p_{true}(A) = (p_{true}(A_1), \ldots, p_{true}(A_n))$ is called the true Poincaré rank of A.

It is shown by Deligne and van den Essen separately in [19, 21], in the multivariate setting that a necessary and sufficient condition for system $[A]$ to be regular singular is that each individual component A_i, considered as a system of ordinary differential equations in x_i, with the remaining variables held as transcendental constants, is regular singular. As a consequence, system $[A]$ is regular singular if and only if its true Poincaré rank is $(0, 0, \ldots, 0)$. To test this regularity, algorithms available for the univariate case of $n = 1$ (e.g. [13, 26]) can be applied separately to each of the individual components. The following corollary follows directly from Theorem 13, showing that the ith component of the true Poincaré rank of system $[A]$ is equal to the true Poincaré rank of the ith associated (univariate) ODS.

Corollary 16. For all $1 \leq i \leq n$ we have

$$
p_{true}(A_i) = p_{true}(\mathcal{A}_i).
$$

Proof. For the proof, it suffices to remark that [2]

$$
p_{true}(\mathcal{A}_i) - 1 < \omega(A_i) = \omega(\mathcal{A}_i) \leq p_{true}(\mathcal{A}_i). \qquad \square
$$

---

[2] Stronger bounds are given in [6, Remark 3]

From this Corollary it does not yet follow for the multivariate case, as in the univariate case, that it is possible to apply a compatible transformation to system $[A]$ such that all the p_i simultaneously equal the $p_{true}(A_i)$. We investigate this possibility in the next section.

In summary, the formal exponential order, the true Poincaré rank, and most importantly the Q_i's in (6), can be obtained efficiently by computations with univariate rather than multivariate series, making use of existing algorithms and packages. As mentioned in the introduction, this exponential part is of central importance in applications since it determines the asymptotic behavior of the solution in the neighborhood of an irregular singularity. To compute a full fundamental matrix of formal solutions, we still have to determine suitable rank reduction transformations. Transformations which reduce the rank of the associated systems do not suffice, since they are not necessarily compatible. We therefore proceed to develop a multivariate rank reduction algorithm.

5. Rank Reduction

In this section, we are interested in the rank reduction of Pfaffian systems, more specifically, the explicit computation of a transformation which, given system $[A]$, yields an equivalent system whose Poincaré rank is the true Poincaré rank. We show that, under certain conditions, the true Poincaré ranks of the subsystems of $[A]$ can be attained simultaneously via a transformation compatible with $[A]$.

We first generalize Moser's reduction criterion [30] to multivariate systems. We then establish an extension of the algorithm we gave in [1] for the bivariate case to the multivariate setting. The main problem in the treatment of multivariate systems is that the entries of the A_i do not necessarily lie in a principal ideal domain. This is a common problem within the study of systems of functional equations. The same obstacle arises in [15] and in the analogous theory of formal decomposition of commuting partial linear difference operators established in [32].

5.1. Generalized Moser's Criterion

For univariate systems, Moser's criterion characterizes systems for which rank reduction is possible. To adapt this criterion to our setting, we follow [7, 30] and define the generalized Moser rank and the Moser invariant of a system $[A]$ as the following n-tuples of rational numbers:

$$m(A) = (m(A_1), \ldots, m(A_n)), \text{where } m(A_i) = \max\left(0, p_i + \frac{\text{rank}(A_{i,0})}{d}\right),$$

$$\mu(A) = (\mu(A_1), \ldots, \mu(A_n)), \text{where } \mu(A_i) = \min(\{m(T(A_i)) \mid T \in GL_d(\mathrm{R}_L)\}).$$

We remark that $\mu(A)$ is well-defined due to Corollary 16.

Definition 17. Consider the partial order $\leq$ on $\mathbb{Q}^n$ for which $\ell < k$ holds if and only if $\ell_j \leq k_j$ for all $1 \leq j \leq n$ and there is at least one index for which the inequality is strict. The system $[A]$ is called reducible if $\mu(A) < m(A)$. Otherwise it is said to be irreducible.

In other words, system $[A]$ is irreducible whenever each of its components is. In particular, it is easy to see from this definition that a system $[A]$ is regular singular if and only if $\mu(A_i) \leq 1$ for all $i \in \{1, \ldots, n\}$, i.e. the true Poincaré rank is a zero n-tuple, which coincides with Deligne's and van den Essen's criterion for regular singular systems.

5.2. Main Theorems

With the help of compatible transformations and the criterion established in Section 5.1, we study the rank reduction of some component A_i of $[A]$ which is given by (2). We will see that rank reduction can be carried out for each component independently without affecting the individual Poincaré ranks of the other components. We fix $i \in \{1, \ldots, n\}$ and we recall that one can expand the components of $[A]$ with respect to x_i. In particular, we have,

$$x_i^{p_i+1} \frac{\partial}{\partial x_i} F = A_i(x)F = (A_{i,0}(\bar{x}_i) + A_{i,1}(\bar{x}_i)x_i + A_{i,2}(\bar{x}_i)x_i^2 + \ldots)F.$$

We set

$$r := \text{rank}(A_{i,0}).$$

For all i we can assume without loss of generality that $A_{i,0}$ is not the zero matrix and thus the reducibility of system $[A]$ coincides with the existence of an equivalent system such that for some i the rank of the leading matrix coefficient $\tilde{A}_{i,0}$ is less than r. We establish the following theorem:

Theorem 18. Consider a Pfaffian system $[A]$ and suppose that $m(A_i) > 1$ for some index $i \in \{1, \ldots, n\}$. If $\mu(A_i) < m(A_i)$ then the polynomial

$$\theta_i(\lambda) := x_i^r \det(\lambda I + \frac{A_{i,0}}{x_i} + A_{i,1})|_{x_i=0} \tag{10}$$

vanishes identically in λ.

Proof. Suppose that there exists a transformation $T(x) \in \mathrm{R}_L^{d \times d}$ which reduces $m(A_i)$ for some $i \in \{1, \ldots, n\}$. That is, setting $T[A] := [\tilde{A}]$, we have:

$$\mathrm{m}(\tilde{A}_i) < \mathrm{m}(A_i). \tag{11}$$

The ith component of system $[A]$ can also be viewed as a system of ordinary differential equations (ODS) in x_i upon considering the x_j's for $j \in \{1, \ldots, n\}$, $j \neq i$, to be transcendental constants. Hence, by (11) and [30, Theorem 1], $\theta_i(\lambda) = 0$. $\square$

Intuitively, the characteristic polynomial of A_i/x_i is used in (10) to detect the true Poincaré rank of the ith component via the valuation of x_i. It turns out that the valuation is only influenced by $A_{i,0}$ and $A_{i,1}$. Even though the true Poincaré rank can be determined from the associated ODS, the criterion is essential as it leads to the construction of the transformation T.

The converse of Theorem 18 also hold true under certain conditions (see Theorem 26): If $\theta_i(\lambda)$ vanishes for some index $i \in \{1, \ldots, n\}$ then we can construct a compatible transformation $T \in GL_d(\mathrm{R}_L)$ which reduces $m(A_i)$. We will establish this result and describe the steps of the algorithm after establishing a series of intermediate ones. We will need two kinds of transformations, shearing transformations and column reductions, which we explain in the next two subsections.

Consider the expansion $A_i = \sum_{k=0}^{\infty} A_{i,k} x_i^k$ of A_i with respect to x_i for a fixed $i \in \{1, \ldots, n\}$. Shearing transformations are polynomial transformations that, roughly speaking, are used to exchange blocks between the $A_{i,k}$'s. The ones we consider here are of the form

$$S = \text{Diag}(x_i^{\beta_1}, \ldots, x_i^{\beta_d}),$$

with $\beta_j \in \{0, 1\}$ for all $j \in \{1, \ldots, d\}$. We illustrate the shearing effect of such a transformation in an easy example.

Example 19. We apply a shearing transformation to a univariate system $[A]$ given by $x_1^{p_1+1} \frac{\partial}{\partial x_1} F = A_1 F$ where $A_1 = \sum_{k=0}^{\infty} A_{1,k} x_1^k$, with

$$A_{1,0} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{1,1} = \begin{pmatrix} 4 & 9 & 2 & -5 \\ 8 & 9 & 0 & 0 \\ 8 & 6 & 2 & 4 \\ 5 & 6 & 3 & 3 \end{pmatrix}.$$

The transformation $S = \text{Diag}(x_1, x_1, 1, 1)$ yields the equivalent system $[\tilde{A}]$ given by

$$x_1^{p_1+1} \frac{\partial}{\partial x_1} F = \tilde{A}_1 F \quad \text{where} \quad \tilde{A}_1 = S^{-1} A_1 S - x_1^p \, \text{Diag}(1, 1, 0, 0).$$

As we are interested in the effect of S on the first few terms of A_1, we look into $S^{-1} A_1 S$ which exchanges the upper right and lower left 2×2 blocks of the A_1 as exhibited in the following diagram:
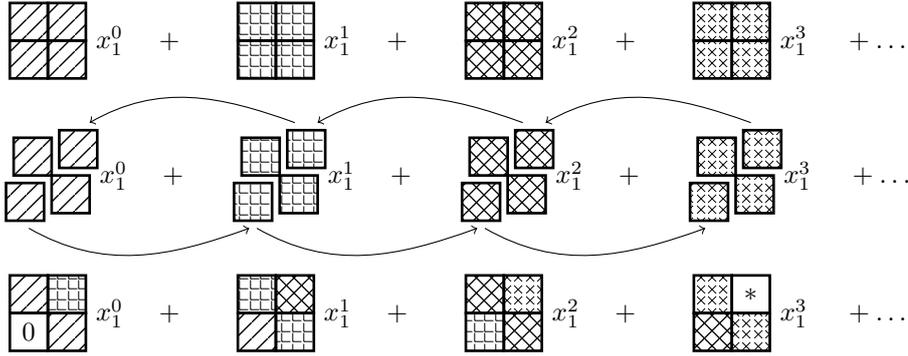
Consequently, $A_{1,0}$ and $A_{1,1}$ become

$$\tilde{A}_{1,0} = \begin{pmatrix} 1 & 2 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{A}_{1,1} = \begin{pmatrix} 4 & 9 & * & * \\ 8 & 9 & * & * \\ -2 & 0 & 2 & 4 \\ 0 & 1 & 3 & 3 \end{pmatrix}.$$

Note that the lower left zero entries in $\tilde{A}_{1,0}$ come from $A_{1,-1}$, which is a zero matrix. In return, the upper right block of $A_{1,0}$ is sent to $A_{1,-1}$. Since it is a zero block, this

transformation does not introduce denominators. The upper right entries in $\tilde{A}_{1,1}$ come from $A_{1,2}$. With this transformation, we reduced the rank of $A_{1,0}$ from 2 to 1.

More generally, let $[A]$ be a multivariate Pfaffian system. The transformation $F = SG$, where S is a shearing transformation in x_i, yields an equivalent system with:

$$\begin{cases} \tilde{A}_i = S^{-1}A_iS - x_i^{p_i} \operatorname{Diag}(\beta_1, \ldots, \beta_d), \\ \tilde{A}_j = S^{-1}A_jS, \quad 1 \le j \ne i \le n, \end{cases}$$

where

$$S^{-1}A_jS = \begin{pmatrix} A_{j,11} & A_{j,12}x_i^{\beta_2-\beta_1} & \ldots & A_{j,1d}x_i^{\beta_d-\beta_1} \\ A_{j,21}x_i^{\beta_1-\beta_2} & A_{j,22} & \ldots & A_{j,2d}x_i^{\beta_d-\beta_2} \\ \vdots & \vdots & \ldots & \vdots \\ A_{j,d1}x_i^{\beta_1-\beta_d} & A_{j,d2}x_i^{\beta_2-\beta_d} & \ldots & A_{j,dd} \end{pmatrix} \quad \text{for all } 1 \le j \le n.$$

The shearing in Example 19 reduced the rank of the leading matrix coefficient and was compatible with the system, i.e. it did not introduce undesired denominators of x_i, because of the column reduced form of $A_{i,0}$. The input system is not always given in such a form for $A_{i,0}$, and so we investigate in the following subsection how to achieve it.

5.4.   Column Reduction

To enable rank reduction, we alternate between the shearing transformation and transformations which reduce some columns of a leading matrix coefficient to zero. For this we discuss in this section the following problem.

(P)   Given a square matrix $A = [v_1, \ldots, v_d] \in \operatorname{Mat}_{d \times d}(\mathrm{R})$ (where v_i denotes the ith column) of rank $r < d$ when considered as an element of $\operatorname{Mat}_{d \times d}(\mathrm{K})$, find a transformation $T \in GL_d(\mathrm{R})$ such that the last $d - r$ columns of TAT^{-1} are zero.

Before considering the algorithmic aspects, we first discuss the existence of such a transformation. As the next example shows, the desired transformation does not necessarily exist for any matrix A.

Example 20. The matrix

$$\begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is obviously of rank 1 over K. There is, however, no transformation $T \in GL_3(\mathrm{R})$ such that TAT^{-1} contains only one non-zero column.

Consider the finitely generated R^d-submodule $M := \langle v_1, \ldots, v_d \rangle$. We call it the column module of A. In order to construct a suitable transformation for bivariate Pfaffian systems (for which the leading matrix coefficients are univariate) the authors of [1] use the fact that $\mathbb{C}[[x_1]]$ and $\mathbb{C}[[x_2]]$ are principal ideal domains and hence that every finitely

generated submodule of a free module over this ring is free. We generalize this for the multivariate case by showing in Corollary 22 that the freeness of the column module M is a necessary and sufficient condition for the existence of a transformation that meets our requirements. This is a direct consequence of Nakayama's Lemma for local rings.

Theorem 21. [29, Theorem 2.3, pp 8] Let R be a local ring, $\mathcal{M}$ its maximal ideal and let M be a finitely generated R-module. Then $v_1, \ldots, v_r \in M$ form a minimal set of generators for M if and only if their images $\bar{v}_1, \ldots, \bar{v}_r$ under the canonical homomorphism $M \to M/\mathcal{M}M$ form a basis of the vector space $M/\mathcal{M}M$ over the field R/$\mathcal{M}$.

The central consequence of Theorem 21 for us is that if M is free, a module basis of M can be chosen among the columns of A. We adapt the theorem to our situation to show that we can bring A into a column-reduced form if and only if its column module is free.

Corollary 22. Let $A \in \text{Mat}_{d \times d}(\text{R})$ be of rank r over K and let M be the module generated by the columns of A. If M is free, then there exists a subset B of the columns in A with r elements such that B is a module basis of M. Furthermore, B is also a K-vector space basis of the column space of A.

Proof. By Theorem 21 we can find a basis $B = \{b_1, \ldots, b_k\}$ of M among the columns of A. By definition, the b_i are linearly independent over R, so they are also linearly independent over K (otherwise, multiplying a linear relation in K with a common denominator yields a relation in R). Since B is a basis of the column module, it also contains a generating set of the K-vector space generated by the columns of A. $\square$

In theory, Corollary 22 would allow the computation of a unimodular column reduction transformation as required in (P) simply via Gaussian elimination. Assume we are given a matrix A and already know a subset $B = (b_1, \ldots, b_r)$ of the columns of A which forms a basis of the column module. Let v be a column vector of A which is not in B. Then, since B is a vector space basis, there exist $c_1, \ldots, c_r \in \text{K}$ such that

$$c_1 b_1 + \cdots + c_r b_r = v.$$

By assumption, B is also a module basis, so there also exist $d_1, \ldots, d_r \in \text{R}$ with

$$d_1 b_1 + \cdots + d_r b_r = v.$$

The b_i are linearly independent, and therefore the cofactors of v with respect to B are unique. It follows that $c_i = d_i$ for all $1 \leq i \leq r$.

The main algorithmic difficulty stems from the fact that not all formal power series admit a finite representation, and even if the initial system is given in a finite form, the splitting transformation as in Theorem 11 does not preserve finiteness. In particular, we face two main problems when working with truncated power series:

$(P1)$ Detecting the correct rank and the linear independent columns of A

$(P2)$ If we know the independent columns, a column reduction transformation computed after truncation is not uniquely determined.

These computational problems arise for general multivariate and for bivariate systems, but were not addressed in previous algorithmic works on this topic [1, 15, 25]. Before we propose our resolution, we illustrate both problems in the following example:

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Example 23. Consider the matrix

$$\begin{pmatrix} x & 0 & x^2 & x^2 + x \\ 0 & x & x & x \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Here, the first three columns v_1, v_2, v_3 are linearly independent and generate the column module. A linear combination of the fourth column v_4 is given by

$$1 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 = v_4.$$

When truncating at order 1, the system is given as

$$\begin{pmatrix} x & 0 & 0 & x \\ 0 & x & x & x \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The original rank cannot be determined from the truncated matrix. Furthermore, even if we know that v_1, v_2, v_3 are linearly independent, there are several linear combinations of the fourth column after truncation:

$$1 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 = v_4.$$

$$1 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 = v_4.$$

The cofactors of the second linear combination are not the truncated cofactors of the first. It can not be extended with higher order terms to a suitable linear combination over the formal power series ring without truncation.

We can solve both $(P1)$ and $(P2)$ with the help of minors of the original system. Let r be the rank of A. Then there exists a nonzero $r \times r$ submatrix B of A whose determinant is nonzero. Let k be the order of the determinant. If we take the truncated system $\tilde{A} = A \operatorname{rem} x^{k+1}$, the same submatrix $\tilde{B}$ in $\tilde{A}$ will have a non-zero determinant modulo x^{k+1} and we can therefore identify in $\tilde{A}$ which columns in A are linearly independent. This resolves $(P1)$ as long as the truncation order k is chosen big enough.

Next assume that for instance the first r columns of A are linearly independent, i.e. we can choose B such that its columns correspond to $v_1, \ldots, v_r$. Let k be as above, ℓ be a positive integer and let v be a column vector that is linearly dependent on the columns of B. Then there exist $c_1, \ldots, c_r \in \mathbb{C}[[x_1, \ldots, x_n]]$ such that

$$B \cdot (c_1, \ldots, c_r) = v.$$

By Cramer's rule, we know that the c_i are given by

$$c_i = \frac{\det(B_i)}{\det(B)}, \tag{12}$$

where B_i is the matrix obtained by replacing the ith column of B by v. Rewriting Equation (12) gives

$$\det(B)c_i - \det(B_i) = 0, \tag{13}$$

and this equation allows the computation of c_i by coefficient comparison. In particular, we are guaranteed to obtain the correct c_i up to order ℓ if in (13) we replace B by $\tilde{B}$, its

truncation at order $\ell + k + 1$, and B_i by $\tilde{B}_i$, the truncation of B_i at order $\ell + k + 1$. This resolves $(P2)$.

This approach is based on the fact that there is a truncation order k such that we can find a submatrix of maximal dimension with non-zero determinant. We have to remark, however, that by the nature of formal power series, it is in general not possible to tell a priori if a given truncation is high enough. Furthermore, we emphasize that it is in general not possible to draw a conclusion about the freeness of the column module from the integral relations among the truncated column vectors, since any linear combination of the form $c_1 v_1 + \cdots + c_{d-1} v_{d-1} - v_d = 0 \mod x^k$ can require a non-unit cofactor c_d for higher truncation orders. However, if no integral relations can be found with the above method, also the column module without truncation cannot be free. Both observations lead to the following practical approach. The full algorithm is carried out with a given truncation order. If the output is correct (compared to the invariant exponential part which can be obtained by Theorem 13), we are done. If not, we increase the truncation order until we get a correct output or arrive at a point where no integral relations can be found anymore. This procedure necessarily terminates, since there exists a suitable truncation order.

One should note that not every K-vector space basis of the column space of A is also a module basis. So, in the worst case, $\binom{d}{r}$ submatrices have to be tested to obtain a module basis.

5.5. Converse of Theorem 18

We consider again a multivariate system $[A]$ as in (2). We fix $i \in \{1, \ldots, n\}$ and investigate the rank reduction of its ith component given by

$$x_i^{p_i+1} \frac{\partial}{\partial x_i} F = A_i F = (A_{i,0} + A_{i,1}x_i + A_{i,2}x_i^2 + A_{i,3}x_i^3 + \ldots)F, \qquad (14)$$

where the matrices $A_{i,j}$ have their entries in $R_{\bar{x}_i}$ and the algebraic rank of $A_{i,0}$ is denoted by r. We recall that we defined $\bar{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $R_{\bar{x}_i} := \mathbb{C}[[\bar{x}_i]]$.

The establishment of the converse of Theorem 18 for the reduction in x_i follows essentially the steps of that of the bivariate case which was given in [1]. The construction requires successive application of transformations in $GL_d(R_{\bar{x}_i})$ and shearing transformations in x_i. We remark that when applying a transformation $T \in GL_d(R_{\bar{x}_i})$ on the ith component, (5) reduces to

$$\tilde{A}_i = T^{-1} A_i T.$$

Given (14), if the column module of $A_{i,0}$ is free, then one can compute a transformation $U_1 \in GL_d(R_{\bar{x}_i})$ such that

$$U_1^{-1} A_{i,0} U_1 = \begin{pmatrix} B^{11} & O \\ B^{21} & O \end{pmatrix}$$

has rank r, entries in $R_{\bar{x}_i}$ and with diagonal blocks of sizes $r \times r$ and $(d-r) \times (d-r)$ respectively. Let v be the rank of B^{11}. If also the column module of B^{11} is free, then one can compute a transformation $U_2 \in GL_r(R_{\bar{x}_i})$ such that

$$U_2^{-1} B^{11} U_2 = \begin{pmatrix} E^{11} & O \\ E^{21} & O \end{pmatrix}$$

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has rank v, entries in $\mathrm{R}_{\bar{x}_i}$ and with diagonal blocks of sizes $v \times v$ and $(r-v) \times (r-v)$ respectively. We set $U := \mathrm{Diag}(U_2, I_{d-r}) \cdot U_1$. Then the leading coefficient $\tilde{A}_{i,0}$ of the equivalent system $U[A_i]$ has the following form:

$$\tilde{A}_{i,0} \;=\; \begin{pmatrix} \tilde{A}_{i,0}^{11} & O & O \\[4pt] \tilde{A}_{i,0}^{21} & O_{r-v} & O \\[4pt] \tilde{A}_{i,0}^{31} & \tilde{A}_{i,0}^{32} & O_{d-r} \end{pmatrix} \tag{15}$$

with diagonal blocks of sizes $v \times v$, $(r-v) \times (r-v)$ and $(d-r) \times (d-r)$ respectively for some $0 \le v < r$ and where

$$\begin{pmatrix} \tilde{A}_{i,0}^{11} \\[4pt] \tilde{A}_{i,0}^{21} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{A}_{i,0}^{11} & O \\[4pt] \tilde{A}_{i,0}^{21} & O \\[4pt] \tilde{A}_{i,0}^{31} & \tilde{A}_{i,0}^{32} \end{pmatrix}$$

are $r \times v$ and $d \times r$ matrices of full column ranks v and r respectively. Clearly, U is compatible with system A since it is unimodular.

From now on, we assume that the leading coefficient $A_{i,0}$ of (14) is in form (15). In particular, we require the column module of $A_{i,0}$ and the column module of B^{11} as given above to be free. We then partition $A_{i,1}$ in accordance with $A_{i,0}$ and set

$$G_{A_i}(\lambda) := \begin{pmatrix} A_{i,0}^{11} & O & A_{i,1}^{13} \\[4pt] A_{i,0}^{21} & O & A_{i,1}^{23} \\[4pt] A_{i,0}^{31} & A_{i,0}^{32} & A_{i,1}^{33} + \lambda I_{d-r} \end{pmatrix}. \tag{16}$$

The polynomial $\det(G_{A_i}(\lambda))$ vanishes identically in λ if and only if $\theta_i(\lambda)$ given by (10) does. In fact, let $D(x_i) = \mathrm{Diag}(x_i I_r, I_{d-r})$. Then we can write $x_i^{-1} A_i(x) = N(x) D^{-1}(x_i)$ where $N(x) \in \mathrm{R}^{d \times d}$, and set $D_0 = D(x_i = 0)$, $N_0 = N(x_i = 0)$. Then we have

$$\begin{aligned} \det(G_{A_i}(\lambda)) &= \det(N_0 + \lambda D_0) = \det(N + \lambda D)|_{x_i=0} \\ &= (\det(\frac{A}{x_i} + \lambda I_d) \det(D))|_{x_i=0} \\ &= (\det(\frac{A_{i,0}}{x_i} + A_{i,1} + \lambda I_d) x_i{}^r)|_{x_i=0} = \theta_i(\lambda). \end{aligned}$$

Moreover, $G_{A_i}(\lambda)$ has an additional important application within the construction of a desired transformation as we show in the following proposition:

Proposition 24. Suppose that $m(A_i) > 1$ and $\det(G_{A_i}(\lambda))$ is identical to zero. If the row module of $G_{A_i}(\lambda = 0)$ is free then there exists a transformation $Q(\bar{x}_i)$ in $GL_d(\mathrm{R}_{\bar{x}_i})$ with $\det(Q) = \pm 1$, compatible with system A, such that the matrix $G_{\tilde{A}_i}(\lambda))$ has the

form

$$G_{\tilde{A}_i}(\lambda) = \begin{pmatrix} A_{i,0}^{11} & O & U_1 & U_2 \\ A_{i,0}^{21} & O & U_3 & U_4 \\ V_1 & V_2 & W_1 + \lambda I_{d-r-\varrho} & W_2 \\ M_1 & O & M_3 & W_3 + \lambda I_\varrho \end{pmatrix},$$ (17)

where $0 \le \varrho \le d - r$, and

$$\mathrm{rank} \begin{pmatrix} A_{i,0}^{11} & U_1 \\ A_{i,0}^{21} & U_3 \\ M_1 & M_3 \end{pmatrix} = \mathrm{rank} \begin{pmatrix} A_{i,0}^{11} & U_1 \\ A_{i,0}^{21} & U_3 \end{pmatrix},$$ (18)

$$\mathrm{rank} \begin{pmatrix} A_{i,0}^{11} & U_1 \\ A_{i,0}^{21} & U_3 \end{pmatrix} < r.$$ (19)

Proof. If the row module of $G_{A_i}(\lambda = 0)$ is free then the transformation $Q(\bar{x}_i)$ can be constructed as in the proof of [1, Proposition 3] for the bivariate case. □

We remark that in the particular case of $v = 0$, (15) is given by

$$\tilde{A}_{i,0}(\bar{x}_i) = \begin{pmatrix} O_r & O \\ \tilde{A}_{i,0}^{32} & O_{d-r} \end{pmatrix}, \quad \text{with} \quad \mathrm{rank}(\tilde{A}_{i,0}^{32}) = r.$$

Consequently, it can be easily verified that (17) is given by

$$G_{\tilde{A}_i}(\lambda) = \begin{pmatrix} O_r & U_3 \\ V_2 & W_1 + \lambda I_{d-r} \end{pmatrix}, \quad \text{and} \quad \varrho = 0.$$

Proposition 25. If $m(A_i) > 1$ and $\det(G_{A_i}(\lambda)) \equiv 0$ is as in (17) with conditions (18) and (19) satisfied, then the component A_i of A in (2) is reducible and reduction can be carried out with the shearing $F = S(x_i) G$ where

$$\begin{cases} S(x_i) = \mathrm{Diag}(x_i I_r, I_{d-r-\varrho}, x_i I_\varrho) & \text{if } \varrho \ne 0 \\ S(x_i) = \mathrm{Diag}(x_i I_r, I_{d-r}) & \text{otherwise.} \end{cases}$$

Furthermore, this shearing is compatible with system A.

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Proof. Given system (2). For any $j \in \{1, \ldots, n\}$ we partition A_j according to (17)

$$A_j = \begin{pmatrix} A_j^{11} & A_j^{12} & A_j^{13} & A_j^{14} \\ A_j^{21} & A_j^{22} & A_j^{23} & A_j^{24} \\ A_j^{31} & A_j^{32} & A_j^{33} & A_j^{34} \\ A_j^{41} & A_j^{42} & A_j^{43} & A_j^{44} \end{pmatrix}, \quad 1 \leq j \leq n,$$

where $A_j^{11}, A_j^{22}, A_j^{33}, A_j^{44}$ are square matrices of dimensions $v, r - v, d - r - \varrho$, and ϱ respectively. It is easy to verify that the equivalent system $S[A] \equiv \tilde{A}$ given by (4) admits the form

$$\tilde{A}_i = \begin{pmatrix} A_i^{11} & A_i^{12} & x_i^{-1}A_i^{13} & A_i^{14} \\ A_i^{21} & A_i^{22} & x_i^{-1}A_i^{23} & A_i^{24} \\ x_i A_i^{31} & x_i A_i^{32} & A_i^{33} & x_i A_i^{34} \\ A_i^{41} & A_i^{42} & x_i^{-1}A_i^{43} & A_i^{44} \end{pmatrix} - x_i^{p_i} \operatorname{Diag}(I_r, O_{d-r-\varrho}, I_\varrho)$$

$$\tilde{A}_j = \begin{pmatrix} A_j^{11} & A_j^{12} & x_j^{-1}A_j^{13} & A_j^{14} \\ A_j^{21} & A_j^{22} & x_i^{-1}A_j^{23} & A_j^{24} \\ x_j A_j^{31} & x_j A_j^{32} & A_j^{33} & x_j A_j^{34} \\ A_j^{41} & A_j^{42} & x_j^{-1}A_j^{43} & A_j^{44} \end{pmatrix}, \quad 1 \leq j \neq i \leq n.$$

Hence, the leading matrix coefficient of the equivalent i^{th}-component is given by

$$\tilde{A}_{i,0}(\bar{x}_i) = \begin{pmatrix} A_{i,0}^{11} & O & U_1 & O \\ A_{i,0}^{21} & O & U_3 & O \\ O & O & O & O \\ M_1 & O & M_3 & O \end{pmatrix}$$

where $\operatorname{rank}(\tilde{A}_{i,0}) < r$ since (18) and (19) are satisfied.

It remains to prove the compatibility of $S(x_i)$ with the system (2), in particular, that the normal crossings are preserved. It suffices to prove that the entries of $A_j, 1 \leq j \neq i \leq n$, which will be multiplied by x_i^{-1} upon applying $S(x_i)$, namely, the entries of A_j^{13}, A_j^{23}, and A_j^{43} are zero matrices modulo x_i otherwise poles in x_i will be introduced. This can be restated as requiring $A_j^{13}(x_i = 0), A_j^{23}(x_i = 0)$, and $A_j^{43}(x_i = 0)$ to be zero submatrices. This requirement is always satisfied due to the integrability condition and the resulting

equality, obtained by setting $x_i = 0$, which we restate here

$$x_j^{p_j+1} \frac{\partial}{\partial x_j} A_{i,0} = A_j(x_i = 0) A_{i,0} - A_{i,0} A_j(x_i = 0), \quad 1 \leq j \neq i \leq n. \tag{20}$$

This equality induces a structure of $A_j(x_i = 0)$ which depends on that of $A_{i,0}$. Since $G_{A_i}(\lambda)$ is as in (17), then, before applying the shearing transformation, $A_{i,0}(\bar{x}_i)$ has the following form (21) and $A_j(x_i = 0)$ can be partitioned accordingly. So we have for $1 \leq j \neq i \leq n$

$$A_{i,0}(\bar{x}_i) = \begin{pmatrix} A_{i,0}^{11} & O & O & O \\ A_{i,0}^{21} & O_{(r-v)(r-v)} & O & O \\ V_1 & V_2 & O_{(d-r-\varrho)(d-r-\varrho)} & O \\ M_1 & O & O & O_{\varrho\varrho} \end{pmatrix}, \tag{21}$$

$$A_j(x_i = 0) = \begin{pmatrix} A_j^{11}(x_i = 0) & A_j^{12}(x_i = 0) & A_j^{13}(x_i = 0) & A_{(j)}^{14}(x_i = 0) \\ A_j^{21}(x_i = 0) & A_j^{22}(x_i = 0) & A_j^{23}(x_i = 0) & A_j^{24}(x_i = 0) \\ A_j^{31}(x_i = 0) & A_j^{32}(x_i = 0) & A_j^{33}(x_i = 0) & A_j^{34}(x_i = 0) \\ A_j^{41}(x_i = 0) & A_j^{42}(x_i = 0) & A_j^{43}(x_i = 0) & A_j^{44}(x_i = 0) \end{pmatrix}. \tag{22}$$

Inserting (21) and (22) in (20), one can obtain the desired results by equating the entries of (20). More explicitly, upon investigating the entries in (Column 3), (Rows 1 and 2, Column 2), and (Row 4, Column 2), we observe the following respectively:

- We have that

$$\begin{pmatrix} A_{i,0}^{11} & O \\ A_{i,0}^{21} & O \\ V_1 & V_2 \\ M_1 & O \end{pmatrix} \cdot \begin{pmatrix} A_j^{13}(x_i = 0) \\ A_j^{23}(x_i = 0) \end{pmatrix} = O_{n,n-r-\varrho}.$$

The former matrix is of full rank r by construction thus $\begin{pmatrix} A_j^{13}(x_i = 0) \\ A_j^{23}(x_i = 0) \end{pmatrix}$ is a zero matrix.

- We also get

$$\begin{pmatrix} A_{i,0}^{11} \\ A_{i,0}^{21} \end{pmatrix} \cdot A_j^{12}(x_i = 0) = O_{r,r-v}.$$

The former is of full rank v by construction thus $A_j^{12}(x_i = 0)$ is a zero matrix.

- Finally, $A_j^{43}(x_i = 0) \cdot V_2 - M_1 \cdot A_j^{12}(x_i = 0) = O_{\varrho,(r-v)}$. Since $A_j^{12}(x_i = 0)$ is null and V_2 is of full column rank $r - v$ by construction then $A_j^{43}(x_i = 0)$ is a zero matrix as well.

This completes the proof. $\square$

We can thus establish the following theorem:

Theorem 26. Consider a Pfaffian system $[A]$ and suppose that $m(A_i) > 1$ for some index $i \in \{1, \ldots, n\}$. If $\theta_i(\lambda)$ given by (10) vanishes for some index $i \in \{1, \ldots, n\}$, then under the conditions required to attain (15) and Proposition 24, we can construct a compatible transformation $T \in GL_d(\mathrm{R}_L)$ which reduces $m(A_i)$ (and consequently $m(A)$). In this case, T can be chosen to be a product of transformations in $GL_d(\mathrm{R}_{\bar{x}_i})$ and polynomial transformations of the form $\mathrm{Diag}(x_i^{\beta_1}, \ldots, x_i^{\beta_d})$ where $\beta_1, \ldots, \beta_d$ are non-negative integers.

Proof. Under the required conditions, we can assume that $A_{i,0}$ has the form (15). Let $G_{A_i}(\lambda)$ be given by (16). Then $\det(G_{A_i}(\lambda))$ vanishes identically in λ if and only if $\theta_i(\lambda)$ does. Then the system $S[Q[A]]$ where S, Q are as in Propositions 24 and 25 respectively, has the desired property. $\square$

For a given index $i \in \{1, \ldots, n\}$, the algebraic rank of the leading matrix coefficient can be decreased as long as $\theta_i(\lambda)$ vanishes identically in λ. In case the leading matrix coefficient eventually reduces to a zero matrix, the Poincaré rank drops at least by one. This process can be repeated until the Moser rank of system $[A]$ equals to its Moser invariant. Due to the compatibility of T in Theorem 18, rank reduction can be applied to any of the components of $[A]$ without altering the Moser rank of the others. Hence, by Corollary 16, the true Poincaré rank of system $[A]$ can be attained by a successive application of the rank reduction to each of its components.

Finally, we remark that the conditions of Theorem 26 are always satisfied in the bivariate case $n = 2$ of arbitrary dimension.

5.6. Examples

Example 27. Consider the completely integrable Pfaffian system with normal crossings given by

$$
\begin{cases}
x_1^2 \frac{\partial}{\partial x_1} F = A_1 F = \begin{pmatrix} (x_1 x_2 x_3 + 1)(x_1 - 1) & x_3(x_1 - 1) \\ x_1 x_2(1 - 2x_1 + x_1 x_2 x_3 - x_1^2 x_2 x_3) & x_1 x_2 x_3(1 - x_1) \end{pmatrix} F, \\
x_2^3 \frac{\partial}{\partial x_2} F = A_2 F = \begin{pmatrix} (2 + 3x_2)(x_1 x_2 x_3 + 1) & x_3(2 + 3x_2) \\ -x_1 x_2(3x_1 x_2^2 x_3 + 2x_1 x_2 x_3 + x_2^2 + 3x_2 + 2) & -x_1 x_2 x_3(2 + 3x_2) \end{pmatrix} F, \\
x_3 \frac{\partial}{\partial x_3} F = A_3 F = \begin{pmatrix} 1 & 0 \\ -x_1 x_2 & 0 \end{pmatrix} F.
\end{cases}
$$

A fundamental matrix of formal solutions is given by:

$$
\Phi(x_1, x_2, x_3) x_1^{C_1} x_2^{C_2} x_3^{C_3} e^{q_1(x_1^{-1/s_1})} e^{q_2(x_2^{-1/s_2})} e^{q_3(x_3^{-1/s_3})}. \tag{23}
$$

If we only seek to compute the exponential parts q_1, q_2, q_3 in (23), then from the associated ODS, we compute:

$$
\begin{cases}
s_1 = 1 \text{ and } q_1(x_1) = \frac{1}{x_1}, \\
s_2 = 1 \text{ and } q_2(x_2) = \frac{-1 - 3x_2}{x_2^2}, \\
s_3 = 1 \text{ and } q_3(x_3) = 0.
\end{cases}
$$

Furthermore, if we wish to compute a fundamental matrix of formal solutions, then following the steps of our formal reduction algorithm, we look at the leading coefficients of each of the three components of the given system. If one of these coefficients has two distinct eigenvalues, the we can apply the splitting lemma (Theorem 11). Indeed, since $A_{1,0}(x_2, x_3)$ has this property, we can compute such a transformation $F = TG$ up to any order. In particular, up to order 10, we compute:

$$T = \begin{pmatrix} 1 & x_1^3 x_2^3 x_3^4 - x_1^2 x_2^2 x_3^3 + x_1 x_2 x_3^2 - x_3 \\ -x_1 x_2 & 1 \end{pmatrix},$$

which yields the following diagonalized system (up to order 10):

$$\begin{cases} x_1^2 \frac{\partial}{\partial x_1} G = A_1 G = \begin{pmatrix} x_1 - 1 & 0 \\ 0 & x_1^2 x_2 x_3 f(x) \end{pmatrix} G, \\ x_2^3 \frac{\partial}{\partial x_2} G = A_2 G = \begin{pmatrix} (2 + 3x_2) & 0 \\ 0 & x_1 x_2^3 x_3 f(x) \end{pmatrix} G, \\ x_3 \frac{\partial}{\partial x_3} G = A_3 G = \begin{pmatrix} 1 & 0 \\ 0 & x_1 x_2 x_3 f(x) \end{pmatrix} G, \end{cases}$$

with $f(x) := (x_1^2 x_2^2 x_3^2 - x_1 x_2 x_3 + 1)$. Hence, the system can be uncoupled into two subsystems of linear scalar equations and integrated to construct G, and consequently, F.

An example of the reduction process for a system in two variables was given in Example 1. However, examples of dimensions two and three do not cover the richness of the techniques presented. So, to illustrate the full process, we treat an example of dimension six. Due to the size of the system and the number of necessary computation steps, we are not able to include it directly in this paper. It is available in several formats at

<center>http://www.mjaroschek.com/pfaffian/</center>

6. Formal Reduction Algorithm

6.1. The Algorithm In Pseudo Code

We now give the full algorithm in pseudo-code and we refer to more detailed descriptions within the article whenever necessary.

Remark 28. Throughout the article, we adopted the field of complex numbers $\mathbb{C}$ as the base field for the simplicity of the presentation. However, any computable commutative field K with $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ can be considered instead. In this case, the restrictions on the extensions of the base field discussed in [6] apply as well and are taken into consideration within our MAPLE implementation.

Given system $[A]$, we discuss the eigenvalues of the leading matrix coefficients $A_{i,0}$, $i \in \{1, \ldots, n\}$, of its n components. If for all of these components uncoupling is unattainable, then we fix $i \in \{1, \ldots, n\}$ and proceed to compute the exponential order $\omega(A_i)$ from the associated ODS. Suppose that $\omega(A_i) = \frac{\ell}{m}$ with ℓ, m coprime positive integers. One can then set $t = x_i^{1/m}$ (re-adjustment of the independent variable), and perform again rank

<center>27</center>

reduction to get an equivalent system whose i^{th} component has Poincaré rank equal to ℓ and leading matrix coefficient with at least m distinct eigenvalues. Consequently, block-diagonalization can be re-applied to uncouple the ith-component. By Section 3.1, this uncoupling results in an uncoupling for the whole system. As mentioned before, this procedure can be repeated until we attain either a scalar system, i.e. a system whose n components are scalar equations, or a system whose Poincaré rank is given by $(0, \ldots, 0)$. The former is trivial and effective algorithms are given for the latter in [25, Chapter 3].

Algorithm 1: fmfs_pfaff(p, A)

Input:	$p = (p_1, \ldots, p_n), A(x) = (A_1, \ldots, A_n)$ of (2).
Output:	A fundamental matrix of formal solutions (6).

$\{C_i\}_{1 \leq i \leq n} \leftarrow O_n$; $\{Q_i\}_{1 \leq i \leq n} \leftarrow O_d$; $\Phi \leftarrow I_d$

WHILE $d \neq 1$ or $p_i > 0$ for some $i \in \{1, \ldots, n\}$ **DO**

  IF $A_{i,0}$ has at least two distinct eigenvalues

    Split system $[A]$ as in Section 3.1; Update Φ

    FMFS_PFAFF $(p, \tilde{A}^{11})$; Update Φ, $\{C_i\}_{1 \leq i \leq n}$, $\{Q_i\}_{1 \leq i \leq n}$

    FMFS_PFAFF $(p, \tilde{A}^{22})$; Update Φ, $\{C_i\}_{1 \leq i \leq n}$, $\{Q_i\}_{1 \leq i \leq n}$

  ELSE IF $A_{i,0}$ has one non-zero eigenvalue

    Update Q_i from the eigenvalues of $A_{i,0}$

    $A(x) \leftarrow$ Follow Section 3.2 ($A_{i,0}$ is now nilpotent)

    FMFS_PFAFF $(p, \tilde{A}(x))$; Update Φ, $\{C_i\}_{1 \leq i \leq n}$, $\{Q_i\}_{1 \leq i \leq n}$

  ELSE

    Apply rank reduction of Section 5; Update Φ; Update p; Update $A_{i,0}$

    IF $p_i > 0$ and $A_{i,0}$ has at least two distinct eigenvalues

      Split system as in Section 3.1; Update Φ

      FMFS_PFAFF $(p, \tilde{A}^{11}(x))$; Update Φ, $\{C_i\}_{1 \leq i \leq n}$, $\{Q_i\}_{1 \leq i \leq n}$

      FMFS_PFAFF $(p, \tilde{A}^{22}(x))$; Update Φ, $\{C_i\}_{1 \leq i \leq n}$, $\{Q_i\}_{1 \leq i \leq n}$

    ELSE IF $A_{i,0}$ has one non-zero eigenvalue

      Update Q_i from the eigenvalues of $A_{i,0}$

      $A(x) \leftarrow$ Follow Section 3.2; ($A_{i,0}$ is now nilpotent)

      FMFS_PFAFF $(p, \tilde{A}(x))$; Update Φ, $\{C_i\}_{1 \leq i \leq n}$, $\{Q_i\}_{1 \leq i \leq n}$

    ELSE

      Follow Section 4 to compute $\omega(A_i) = \frac{\ell}{m}$

      $x_i \leftarrow x_i{}^m$

      Apply rank reduction of Section 5

      Update Φ; Update p ($p_i \leftarrow \ell$); Update $A_{i,0}$

      Update Q_i from the eigenvalues of $A_{i,0}$

      $A(x) \leftarrow$ Follow Section 3.2; ($A_{i,0}$ is now nilpotent)

      FMFS_PFAFF $(p, A(x))$; Update Φ, $\{C_i\}_{1 \leq i \leq n}$, $\{Q_i\}_{1 \leq i \leq n}$

    END IF

  END IF

END WHILE

RETURN p, A, Φ, $\{C_i\}_{1 \leq i \leq n}$, $\{Q_i\}_{1 \leq i \leq n}$.

Algorithm 2: rankReduce(p, A)

Input:	$p_1 \ldots, p_n, A_1 \ldots, A_n$ of (2).
Output:	$T(x) \in GL_d(\mathrm{R}_L)$ and an irreducible equivalent system $\{T(A)\}$ whose Poincaré rank is its true Poincaré rank and the rank of its leading coefficient matrices is minimal $(\mu(T(A)) = m(T(A)))$

$T \leftarrow I_d$
FOR every i from 1 to n **DO**
  $T_i \leftarrow I_d$; $p_i \leftarrow$ Poincaré rank of A_i
  $U(\bar{x}_i) \leftarrow$ yields the form (15)
  IF $U(\bar{x}_i)$ cannot be determined
    RETURN ERROR "Column module not free." **END IF**
  $A_i \leftarrow U^{-1}A_iU$; $T_i \leftarrow T_iU$
  WHILE $\det(G_{A_i}(\lambda)) = 0$ and $p_i > 0$ **DO**
    $Q(\bar{x}_i), \varrho \leftarrow$ Proposition 24
    IF $Q(\bar{x}_i)$ cannot be determined
      RETURN ERROR "Row module not free." **END IF**
    $S(x_i) \leftarrow$ Proposition 25
    $P \leftarrow QS$; $T_i \leftarrow T_iP$
    $A_i \leftarrow P^{-1}A_iP - x_i^{p_i}S^{-1}\frac{\partial S}{\partial x_i}$
    $p_i \leftarrow$ Poincaré rank of A_i
    $U(\bar{x}_i) \leftarrow$ yields the form (15)
    $A_i \leftarrow U^{-1}A_iU$; $T_i \leftarrow T_iU$
  END WHILE
  FOR every $j \neq i$ from 1 to n **DO**
    $A_j \leftarrow T_i^{-1}A_jT_i - x_j^{p_j+1}T_i^{-1}\frac{\partial}{\partial x_j}T_i$
  END FOR
  $T \leftarrow TT_i$
END FOR
RETURN $(T, p_1, \ldots, p_n, A_1, \ldots, A_n)$.

6.2. An Alternative Rank Reduction Algorithm

In the case of univariate systems, Levelt's investigations of the existence of stationary sequences of free lattices lead to an algorithm which reduces the Poincaré rank to its minimal integer value [26]. This algorithm was then generalized to the bivariate case by the first author of this paper et al. in [15]. The theoretical basis of this algorithm differs substantially from the algorithm given herein based on Moser's criterion. The final result of both approaches however, i.e. the algorithm itself, is based on applying column reductions and shearing transformations in both algorithms, though in a different manner. In fact, the algorithms coincide for the particular case of $\varrho = 0$. The limitation in [15] within the generalization to the multivariate case is in guaranteeing the freeness conditions for the leading matrix coefficient $A_{i,0}$ as stated in Section 5.5. The additional condition of the freeness of the row module as in Proposition 24 is not required. Since the linear algebra problem is resolved in Section 5.4, this results in Algorithm 3.

Although both algorithms have an identical cost [25, pp 108], experimental results for the univariate case and certain bivariate systems (singularly-perturbed linear differential

systems) suggest that the lattice-based algorithm complicates dramatically the coefficients of the system under reduction, even if Moser's criterion is adjoined to avoid some unnecessary computations [2, Section 4.3]. Hence, Algorithm 2 can be used as long as the required freeness conditions hold. Nevertheless, if the freeness of the row module of $G_{A_i}(\lambda = 0)$ is not satisfied, then Algorithm 3 can be used as long as the column modules of $A_{i,0}$ and B^{11} is free. There remains however, the question on the equivalence of these conditions.

Algorithm 3: rankReduce_alt(p, A)

Input:	$p_1 \ldots, p_n, A_1 \ldots, A_n$ of (2).
Output:	$T(x) \in GL_d(\mathrm{R}_L)$ and an irreducible equivalent system $\{T(A)\}$ whose Poincaré rank is its true Poincaré rank and the rank of its leading coefficient matrices is minimal $(\mu(T(A)) = m(T(A)))$

$T \leftarrow I_d$
FOR every i from 1 to n **DO**
   $T_i \leftarrow I_d$; $p_i \leftarrow$ Poincaré rank of A_i
   WHILE $j < d - 1$ and $p_i > 0$ **DO**
     $U(\bar{x}_i) \leftarrow$ yields the form (15)
     IF $U(\bar{x}_i)$ cannot be determined
       RETURN ERROR "Column module not free." **END IF**
     $r = \mathrm{rank}(A_{i,0})$
     $S(x_i) \leftarrow$ Proposition 25 with $\varrho = 0$ (i.e. $S(x_i) \leftarrow \mathrm{Diag}(x_i I_r, I_{d-r})$)
     $P \leftarrow US$
     $A_i \leftarrow P^{-1} A_i P - x_i^{p_i} S^{-1} \frac{\partial S}{\partial x_i}$
     $\tilde{p}_i \leftarrow$ Poincaré rank of A_i
     IF $\tilde{p}_i < p_i$ **THEN**
       $j \leftarrow 0$
     ELSE
       $j \leftarrow j + 1$
     END IF
     $p_i \leftarrow \tilde{p}_i$
     $T_i \leftarrow T_i P$
   END WHILE
   FOR every $j \neq i$ from 1 to n **DO**
     $A_j \leftarrow T_i^{-1} A_j T_i - x_j^{p_j+1} T_i^{-1} \frac{\partial T_i}{\partial x_j}$
   END FOR
   $T \leftarrow T T_i$
END FOR
RETURN $(T, A_{i\{1 \leq i \leq n\}})$.

---

7. Conclusion

In this article, we studied completely integrable Pfaffian systems with normal crossings in several variables. We showed that one can associate a set of univariate linear singular differential systems from which the formal invariants of the former can be retrieved. This

reduces computations to computations over a univariate field via ISOLDE or LINDALG, and limits the numbers of coefficients necessary for the computations. We then complemented our work with a rank reduction algorithm based on generalizing Moser's criterion and the algorithm given by Barkatou in [7]. The former is applicable to any bivariate system. However, for multivariate systems, it demands that several explicitly described conditions are met.

One field of investigation is the possibility of weakening the conditions required in the multivariate setting for the rank reduction. Another one is the adaptation of the techniques developed herein for rank reduction to generalize the notion of *simple* systems (see [11] for $m = 1$). This notion, in the univariate case, gives another approach to construct a basis of the space of regular solutions [12], and the obstacles encountered are similar to those in rank reduction. An additional field is to study closed form solutions [10] in the light of associated ODS introduced in Section 4.

In future work we aim to thoroughly study the theoretical complexity of the proposed algorithm as well as give detailed information of the effects on the truncation of the input system during the computation. This work is not straightforward and even in the case of regular systems, it is not studied in the existing work (i.e. [25, Chapter 3] and references therein).

Systems arising from applications do not necessarily or directly fall into the class of completely integrable Pfaffian systems with normal crossings. Investigations in more general classes can be found in [31, 24, 33] and references therein.

An additional field of investigation is the formal reduction in the difference case using the approaches proposed herein. Praagman established in [32] a formal decomposition of m commuting partial linear difference operators. This study was intended as an analog to that established by Levelt, van den Essen, Gérard, Charrière, Deligne, and others [17, 18, 19, 21].

Acknowledgments

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